QUESTION

(a) Show that $\mathcal{H}_0\left(\frac{1}{r}\right) = \frac{1}{a}$ by using the fact that the Hankel transform is its own inverse.

(b) Solve the cylindrical wave equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

with the boundary conditions

$$\phi(r,0) = \frac{1}{r},\ \ \phi_t(r,0) = 0,$$

$$\phi \rightarrow 0 \text{ as either } r \rightarrow \infty \text{or } t \rightarrow \infty,\ \ \phi_r \rightarrow 0 \text{ as } r \rightarrow \infty.$$

You can leave your final answer as an (inverse Hankel transform) integral.

ANSWER

(a)

$$\mathcal{H}_0\left(\frac{1}{r}\right) = \int_0^\infty \frac{1}{r}\mathcal{J}_0(\alpha r)r\,dr = \int_0^\infty \mathcal{J}_0(\alpha r)\,dr = \frac{1}{\alpha}\int_0^\infty \mathcal{J}_0(x)\,dx = \frac{k}{\alpha}$$

for some constant k. But as the Hankel transform is self inverse, $k = 1$.

(b)

$$\frac{1}{r}(r\phi_r)_r = \frac{1}{c^2}\phi_{tt},\ \ \phi(r,0) = \frac{1}{r},\ \ \phi_t(r,0) = 0,$$

$\phi \rightarrow 0$ as $r \rightarrow \infty$ or $t \rightarrow \infty$, $phi_r \rightarrow 0$ as $r \rightarrow \infty$

Take a Hankel transform of order zero with respect to r

$$-\alpha^2\Phi(\alpha,t) = \frac{1}{c^2}\Phi_{tt}(\alpha,t)$$

$$\begin{aligned}
\Phi(\alpha,t) &= A(\alpha)\cos c\alpha t + B(\alpha)\sin c\alpha t \\
\phi_t(r,0) &= 0\forall r \Rightarrow \Phi_t(\alpha,0) = 0 \ \text{ forall } \alpha \Rightarrow B(\alpha) = 0 \\
\phi(r,0) &= \frac{1}{R} \Rightarrow \Phi(\alpha,0) = \frac{1}{\alpha} \Rightarrow A(\alpha) = \frac{1}{\alpha}
\end{aligned}$$

Inverse Hankel transform

$$\phi(r,t) = \int_0^\infty \frac{1}{\alpha}\cos c\alpha t\mathcal{J}_0(\alpha r)\alpha\,d\alpha = \int_0^\infty \cos c\alpha t\mathcal{J}_0(\alpha r)\,d\alpha$$

Could look this up as a cosine transform.