QUESTION

Use the cosine transform to solve the heat equation

$$U_{xx} = cU_t,\ x > 0,\ 0 < t < \infty,$$

with the boundary conditions

$$U_x(0,t) = 0,\ U(x,0) = e^{-\frac{x^2}{2}},\ U, U_x \to 0 \text{ as } x \to \infty.$$

ANSWER

$U_{xx} = cU_t,\ x > 0,\ 0 < t < \infty,\ U_x(0,t) = 0,$

$U(x,0) = e^{-\frac{X^2}{2}},\ U, U_x \to 0$ as $x \to \infty$

Fourier cosine transform with respect to x gives $U(\xi, t)$

$-U_x(0,t) - \xi^2 u = -\xi^2 u = cu_t$

$\Rightarrow u(\xi,t) = u(\xi,0)e^{-\frac{\xi^2}{c}t}$

$$\begin{aligned}
u(\xi,0) &= \int_0^\infty e^{-\frac{x^2}{2}} \cos \xi x\, dx \\
&= \frac{1}{2}\int_{-\infty}^\infty e^{-\frac{x^2}{2}} \cos \xi x\, dx \\
&= \frac{1}{2}\text{Re}\int_{-\infty}^\infty e^{-\frac{x^2}{2}} e^{-i\xi x}\, dx \\
&= \frac{1}{2}\int_{-\infty}^\infty e^{-\frac{1}{2}(x+i\xi)^2 - \frac{\xi^2}{2}}\, dx \\
&= \frac{1}{2}e^{-\frac{\xi^2}{2}}\int_{-\infty}^\infty e^{-\frac{1}{2}y^2}\, dy \quad (\text{taking } y = x + i\xi) \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{-\xi^2}{2}} \\
\Rightarrow u(\xi,t) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}\left(1+2\frac{t}{c}\right)}
\end{aligned}$$

$U(x,t) = \frac{2}{\pi}\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{\xi^2}{2}(1+2\frac{t}{c})} \cos \xi x\, d\xi$

Change of variable $\overline{\xi} = \sqrt{1+2\frac{t}{c}}\xi,\ \overline{x} = \frac{x}{\sqrt{1+2\frac{t}{c}}}$

$$\begin{aligned}
U(x,t) &= \frac{1}{\sqrt{1+2\frac{t}{c}}}\frac{2}{\pi}\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{\xi^{-2}}{2}} \cos \overline{\xi}\overline{x}\, d\overline{\xi} \\
&= \frac{1}{\sqrt{1+2\frac{t}{c}}}e^{-\frac{\overline{x}^2}{2}} \\
&= \frac{1}{\sqrt{1+2\frac{t}{c}}}e^{-\frac{x^2}{2}\left(1+2\frac{t}{c}\right)}
\end{aligned}$$