QUESTION

Find the sine transform $F_s(\xi), \xi > 0$ of

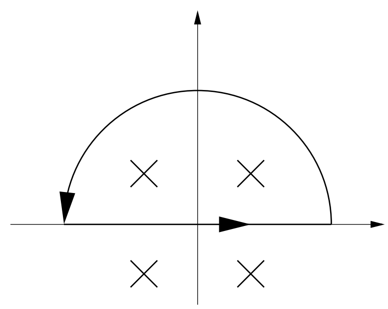
$$f(x) = \frac{x}{1+x^4}$$

by integrating $\dfrac{ze^{i\xi z}}{(1+z^4)}$ around a large semicircle.

ANSWER

$$f(x) = \frac{x}{1+x^4},\ F_s(\xi) = \int_0^\infty \frac{x}{1+x^4} \sin \xi x = \frac{1}{2} \int_{-\infty}^\infty \frac{xe^{i\xi x}}{1+x^4}\, dx$$

For $\xi > 0$ closed contour is in the upper half plane.



$\text{Res}(\dfrac{ze^{i\xi z}}{1+z^4}, z_0) = \dfrac{z_0 e^{i\xi z_0}}{4x_0^3} = -\dfrac{z_0^2}{4} e^{i\xi z_0}$

$$\begin{aligned} F_s(\xi) &= \frac{1}{2}\text{Im } 2\pi i \left(-\frac{i}{4} e^{i\xi \frac{1+i}{\sqrt{2}}} - \frac{-i}{4} e^{i\xi \frac{-1+i}{\sqrt{2}}} \right) \\ &= \frac{\pi}{2}\text{Re } e^{-\frac{\xi}{\sqrt{2}}} \left(\frac{e^{i\xi}\sqrt{2}}{2i} - \frac{e^{-\frac{i\xi}{\sqrt{2}}}}{2i} - \frac{e^{-\frac{i\xi}{\sqrt{2}}}}{2i} \right) \\ &= \frac{\pi}{2} e^{-\frac{\xi}{\sqrt{2}}} \sin \frac{\xi}{\sqrt{2}} \text{ for } \xi > 0 \end{aligned}$$

$F_s(-\xi) = \overline{F_s(\xi)} = F_s(\xi)$

So for any ξ, $\quad F_s(\xi) = \frac{\pi}{2} e^{-\frac{|\xi|}{\sqrt{2}}} \sin \frac{|\xi|}{\sqrt{2}}$