Question

Let F be a closed set. Let A_0 be a set disjoint from F.

Let $A_n = \{x | x \epsilon A_0, \ d(x, F) \geq \frac{1}{n}\}$.

Show that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_0$, and deduce the existence of $\lim_{n\to\infty} m^*(A_n)$ (it may be $+\infty$).

Prove that $A_0 = \bigcup_{n=1}^{\infty} \qquad (\dagger)$

Write $D_n = A_{n+1} - A_n$. Show that, provided $m \geq n+2$,

$d(D_m, D_n) \geq \dfrac{1}{m(n+1)} > 0$.

Consider the sums $\sum_{k=1}^{\infty} m^*(D_{2k})$, $\sum_{k=0}^{\infty} m^*(D_{2k+1})$. If the first is infinite, prove that $m^*(A_{2n}) \to \infty$, and if the second is infinite, prove that $m^*(A_{2n+1}) = +\infty$. Deduce that if either sum is infinite then $\lim m^*(A_n) = +\infty \geq m^*(A_0)$. If both sums are finite, use $(\dagger)$ and $A_0 = A_{2n} \cup \bigcup_{k=n}^{\infty} D_{2k} \cup \bigcup_{k=n}^{\infty} D_{2k+1}$ to show that $m^*(A_0) \leq \lim m^*(A_{2n})$.

Deduce finally that $m^*(A_n) \to m^*(A_0)$ as $n \to \infty$. (None of the A's need be measurable). Use this result to prove that a closed set S is measurable. (Hint: in the definition of measurability, let $E - S = A_0$).

Answer

$A_n = \{x | x \epsilon A_0, d(x, F) \geq \frac{1}{n}\} \subseteq A_0$

If $x \epsilon A_n$ then $x \epsilon A_0$ and $d(x, F) \geq \dfrac{1}{n}$

$\Rightarrow x \epsilon A_0$ and $d(x, F) \geq \dfrac{1}{n+1} \Rightarrow x \epsilon A_{n+1}$

Therefore $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_0$

Hence $m^*(A_n)$ is an increasing sequence. Thus $\lim m^*(A_n) \leq m^*(A_0)$ (may be $+\infty$)

Suppose $x \epsilon D_m$, $\ y \epsilon D_n$. Let $f \epsilon F$

$d(y, x) \geq d(y, f) - d(x, f)$

$\qquad \geq \dfrac{1}{n+1} - d(x, f)$ since $y \epsilon D_n \subseteq A_{n+1}$

Now $x \epsilon D_m$ so $x \epsilon A_{m+1} - A_m$

Hence $\dfrac{1}{m+1} \leq d(x, f) < \dfrac{1}{m}$

Thus $d(y,x) \geq \dfrac{1}{n+1} - \dfrac{1}{m} = \dfrac{m-n-1}{m(n+1)} \geq \dfrac{1}{m(n+1)}$ (1)

provided $m \geq n+2$

Consider the sums $\sum_{k=1}^{\infty} m^*(D_{2k})$, and $\sum_{k=1}^{\infty} m^*(D_{2k+1})$

Suppose $\sum_{k=1}^{\infty} m^*(D_{2k}) = +\infty$

Then $A_{2n} = A_1 \cup \bigcup_{k=1}^{n-1} D_{2k} \cup \bigcup_{k=0}^{n-1} D_{2k+1} \supseteq \bigcup_{k=1}^{n-1} D_{2k}$

Thus $m^*(A_{2n}) \geq m^*\left(\bigcup_{k=1}^{n-1} D_{2k}\right) = \sum_{k=1}^{n-1} m^*(D_{2k})$ by (1) and

$m^*\left(\bigcup_{i=1}^{n} S_i\right) = \sum_{i=1}^{n} m^*(S_i)$

Therefore $m^*(A_{2n}) \to +\infty$ as $n \to \infty$

Similarly if the other sum is $+\infty$ then $m^*(A_{2k+1}) \to +\infty$.

Hence if either sum is $+\infty$, $m^*(A_n) \to +\infty$ as $n \to \infty$

and so $m^*(A_n) \to m^*(A_0) = +\infty$ as $n \to \infty$.

If both sums are convergent then $A_0 = A_{2n} \cup \bigcup_{k=n}^{\infty} D_{2k} \cup \bigcup_{k=n}^{\infty} D_{2k+1}$ and so

$$m^*(A_0) \leq m^*(A_{2n}) + m^*\left(\bigcup_{k=n}^{\infty} D_{2k}\right) + m^*\left(\bigcup_{k=n}^{\infty} D_{2k+1}\right)$$

$$\leq m^*(A_{2n}) + \left(\sum_{k=n}^{\infty} m^*(D_{2k})\right) + \sum_{k=n}^{\infty} m^*(D_{2k+1})$$

Since both sums converge, both sums from n to ∞ tend to zero as $n \to \infty$.

Hence $m^*(A_0) \leq \lim m^*(A_{2n}) = \lim m^*(A_n)$ by monotonicity.

Hence $m^*(A_0) = \lim m^*(A_n)$

Let F be a closed set. Let T be any set, let $A_0 = T - F$

Let $A_n = \{x | x \epsilon A_0, d(x,F) \geq \dfrac{1}{n}\}$

Then $d(A_n, T \cap F) > 0$ and so by theorem 2.8

$m^*(T) = m^*((T \cap F) \cup (T - F))$

$\geq m^*((T \cap F) \cup A_n)$ (since $A_n \subseteq T - F$)

$= m^*(T \cap F) + m^*(A_n)$ for all n.

Therefore $m^*(T) \geq m^*(T \cap F) + \lim m^*(A_n)$

$= m^*(T \cap F) + m^*(T - F)$

Hence the result.