Question

Suppose that $\{S_1, \cdots S_n\}$ is a collection of sets, not necessarily measurable, satisfying $d(S_i, S_j) > 0$ for $i \neq j$. Extend theorem 2.8 to show that

$$m^* \left(\bigcup_{i=1}^{n} S_i \right) = \sum_{i=1}^{n} m^*(S_i)$$

Answer

The result is true for $n = 2$ by theorem 2.8. Suppose true for $n = k$, now consider S_{k+1}. Let $d_i = d(S_i, S_{k+1}) > 0$. Let $d = \min_{1 \leq i \leq k} d_i > 0$. Then

$d(S_{k+1}, \bigcup_{i=1}^{k}) = d > 0$. Hence by theorem 2.8

$m^*(S_{k+1} \cup \bigcup_{i=1}^{k} S_i) = m^*(S_{k+1}) + \sum_{i=1}^{k} S_i$. Hence the result.