Question

Suppose we have an increasing sequence of sets $A_1 \subseteq A_2 \subseteq \cdots$

Let $A_0 = \bigcup_{n=1}^{\infty} A_n$. Let $D_n = A_{n+1} - A_n$. Prove the following identities

i) $A_{2n} = A_1 \cup \bigcup_{k=1}^{n-1} D_{2k} \cup \bigcup_{k=0}^{n-1} D_{2k+1}$

ii) $A_0 = A_{2n} \cup \bigcup_{k=n}^{\infty} D_{2k} \cup \bigcup_{k=n}^{\infty} D_{2k+1}$

Answer

i) $A_{2n} = A_1 \cup \bigcup_{k=1}^{n-1} D_{2k} \cup \bigcup_{k=0}^{n-1} D_{2k+1}$

$n = 2 \quad A_1 \cup D_2 \cup D_1 \cup D_3$

$= A_1 \cup (A_2 - A_1) \cup (A_3 - A_2) \cup (A_4 - A_3) = A_4$

since $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$

$A_1 \cup \bigcup_{k=1}^{n} D_{2k} \cup \bigcup_{k=0}^{n} D_{2k+1} = A_{2n} \cup D_{2n} \cup D_{2n+1}$

$= A_{2n} \cup (A_{2n+1} - A_{2n}) \cup (A_{2n+2} - A_{2n+1}) = A_{2n+2}$

Hence the result by induction.

ii) $A_{2n} \cup \bigcup_{k=n}^{\infty} D_{2k} \cup \bigcup_{k=n}^{\infty} D_{2k+1}$

$= A_1 \cup \bigcup_{k=1}^{n-1} D_{2k} \cup \bigcup_{k=0}^{n-1} D_{2k+1} \cup \bigcup_{k=n}^{\infty} D_{2k} \cup \bigcup_{k=n}^{\infty} D_{2k+1}$

$= A_1 \cup \bigcup_{i=1}^{\infty} D_i = \bigcup_{n=1}^{\infty} A_i = A_0$ since $A_i \subseteq A_{i-1} \cdots$