QUESTION

(a) State the Duality Theorem of linear programming and use it to prove the Theorem of Complementary Slackness.

(b) Use duality theory to determine whether $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 4$, is an optimal solution of the linear programming problem

$$\begin{array}{ll} \text{maximize} & z = 4x_1 + x_2 + 7x_3 + 9x_4 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \\ & 6x_1 + 8x_2 + 3x_3 + x_4 \leq 15 \\ & 3x_1 + 2x_2 + 7x_3 + 4x_4 \leq 18 \\ & 5x_1 + 5x_2 + 8x_3 + 3x_4 = 17. \end{array}$$

(c) For the linear programming problem

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & x_j \geq 0 & j = 1, \ldots, n \\ & \sum\limits_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \ldots, m, \end{array}$$

the optimal value of the objective function is z^* and $y_1^*, \ldots, y_m^*$ are optimal values of the dual variables. Let z^{**} denote the optimal value of the objective function for the linear programming problem

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & x_j \geq 0 & j = 1, \ldots, n \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i + \delta_i & i = 1, \ldots, m. \end{array}$$

$$z^{**} \leq z^* + \sum_{i=1}^{m} \delta_i y_i^*.$$

You may use the Duality Theorem in your proof.

ANSWER

(a) The duality theorem states that

- if the primal problem has an optimal solution, then so has the dual, and $z_p = z_D$;
- if the primal problem is unbounded, then the dual is infeasible;
- if the primal problem is infeasible, then the dual is either infeasible or unbounded.

Consider the following primal and dual problems

Maximize	$z_P = \mathbf{c}^T\mathbf{x}$	Minimize	$z_D = \mathbf{b}^T\mathbf{y}$
subject to	$\mathbf{x} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0}$	subject to	$\mathbf{y} \geq \mathbf{0}, \mathbf{t} \geq \mathbf{0}$
	$A\mathbf{x} + \mathbf{s} = \mathbf{b}$		$A^T\mathbf{y} - \mathbf{t} = \mathbf{c}$

For feasible solutions of the primal and dual, we have

$$Z_P = \mathbf{c}^T\mathbf{x} = (\mathbf{y}^T A - \mathbf{t}^T)\mathbf{x} = \mathbf{y}^T(\mathbf{b} - \mathbf{s}) - \mathbf{t}^T x = z_D - \mathbf{y}^T\mathbf{s} - \mathbf{t}^T\mathbf{x}$$

For an optimal solution of the primal and dual, $z_p = z_D$ so

$$\mathbf{y}^T\mathbf{s} + \mathbf{t}^T\mathbf{x} = 0$$

Since variables are non negative this implies that

$$y_i s_i = 0 \ i = 1, \dots m$$

$$t_j x_j = 0 \ j = 1, \dots, n$$

(b) The solution $x_1 = 0,\ x_2 = 1,\ x_3 = 0,\ x_4 = 4$ yeilds $z = 37\ s_1 = 3,\ s_2 = 0$.

The dual problem is

minimize	$z_D = 15y_1 + 18y_2 + 17y_3$
subject to	$y_1 \geq 0,\ y_2 \geq 0,$
	$6y_1 + 3y_2 + 5y_3 \geq 4$
	$8y_1 + 3y_2 + 5y_3 \geq 1$
$3y_1 + 7y_2 + 8y_3 \geq 7$	
	$y_1 + 4y_2 + 3y_3 \geq 9$

If the given solution is optimal, then we can use the complementary slackness conditions.

$y_1 s_1 = 0$ implies $y_1 = 0$

$x_2 t_2 = 0$ implies $t_2 = 0$

$x_4 t_4 = 0$ implies $t_4 = 0$

Thus,

$$\begin{aligned} 2y_2 + 5y_3 &= 1 \\ 4y_2 + 3y_3 &= 9 \end{aligned}$$

so

$$y_2 = 3,\ y_3 = -1$$

and

$$t_1 = 0,\ t_3 = 6$$

Therefore, the solution is feasible.

Since $z_D = 37 = z$ the proposed solution is optimal.

(c) Using matrix notation, the relevant problems are

$$\begin{array}{lll} \text{(P)} & \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ & \text{subject to} & \mathbf{x} \geq 0 \\ & & A\mathbf{x} \leq \mathbf{b} \end{array} \qquad \begin{array}{lll} \text{(P)} & \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ & \text{subject to} & \mathbf{x} \geq \mathbf{0} \\ & & A\mathbf{x} \leq \mathbf{b} + \delta \end{array}$$

and the dual of (P) is

$$\begin{array}{lll} \text{(D)} & \text{Minimize} & \mathbf{b}^T\mathbf{y} \\ & \text{subject to} & \mathbf{y} \geq \mathbf{0} \\ & & A^T\mathbf{y} \geq \mathbf{c} \end{array}$$

From the duality theorem,

$$z* = \mathbf{b}^T\mathbf{y}* = (\mathbf{y}*)\mathbf{b}$$

Let $\mathbf{x} = \mathbf{x}*$ be an optimal solution of (P’). Then

$$x ** = \mathbf{c}^T\mathbf{x}* \leq (y*)^T Ax* \leq (y*)^T(\mathbf{b} + \delta) = z * + \sum_{i=1}^{m} s_i y_i *$$