QUESTION

(a) Consider the linear programming problem

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & x_j \geq 0 & j = 1, \ldots, n \\ & \sum_{j=1}^{n} a_{ij} x_j = b_i & i = 1, \ldots, m, \end{array}$$

where the constraint matrix $A = (a_{ij})$ has rank m, and $m < n$. Explain briefly what is meant by a *basic feasible solution* of this problem. Prove that an extreme point of the convex set of feasible solutions is a basic feasible solution.

(b) Give a brief explanation of the term *cycling* in the simplex method, and describe two methods by which cycling can be avoided. Explain briefly why the simplex method terminates after a finite number of iterations when cycling does not occur.

(c) **(i)** Solve the following linear programming problem using the dual sim plex method.

$$\begin{array}{ll} \text{Minimize} & z = 5x_1 + 2x_2 + 5x_3 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \\ & 3x_1 + x_2 + 2x_3 \geq 4 \\ & 6x_1 + 3x_2 + 5x_3 \geq 10. \end{array}$$

(ii) Suppose that the additional constraint

$$x_2 + 2x_3 \geq 3$$

is imposed. Obtain an optimal solution for this modified problem.

ANSWER

(a) Let $\mathbf{x}_0^T = (x_1^0, \ldots, x_r^0, 0, \ldots, 0)$ be an extreme point in which the first r components are positive, where $r > m$. Let $\mathbf{a}_1, \ldots \mathbf{a}_r$ be the first r columns of A. Since $r > m$, $\mathbf{a}_1, \ldots \mathbf{a}_r$ are linearly dependent, i.e. there exist $\lambda_1, \ldots, \lambda_r$, not all zero such that

$$\lambda \mathbf{a}_1 + \ldots + \lambda_r \mathbf{a}_r = \mathbf{0}$$

Let

$$\epsilon = min \left\{ \frac{x_j^0}{|\lambda_j|} : \lambda_j \neq 0,\ j = 1, \ldots, r \right\}$$

and consider $\mathbf{x}_0 + \epsilon\lambda,\ \mathbf{x}_2 = \mathbf{x}_0 - \epsilon\lambda$, where $\lambda^T = (\lambda_1, \ldots \lambda_r, 0, \ldots, 0)$. The definition of ϵ ensures that $\mathbf{x}_1 \geq \mathbf{0}$ and $\mathbf{x}_2 \geq \mathbf{0}$. Furthermore,

$$A\mathbf{x}_1 = a(\mathbf{x}|_0 + \epsilon\lambda) = \mathbf{b}\ A\mathbf{x}_2 = A(\mathbf{x}_0 - \epsilon\lambda) = \mathbf{b}$$

so $\mathbf{x}_1$ and $\mathbf{x}_2$ are feasible solutions. However, $\mathbf{x}_0\frac{(\mathbf{x}_1+\mathbf{x}_2)}{2}$ which is impossible if $\mathbf{x}_0$ is an extreme point. Thus $r \leq m$. Also, $\mathbf{a}_1, \ldots \mathbf{a})_r$ are linearly independent, or else the above argument shows that $\mathbf{x}_0$ is not an extreme point. If $r = m$, then $\mathbf{x}_0$ is a basic feasible solution. If $r < m$, it is possible to select $m - r$ columns of A which, together with $\mathbf{a}_1, \ldots \mathbf{a}_r$ are linearly independent. The corresponding variables define a basic feasible solution.

(b) The simplex method cycles if a sequence of tableaus keeps reappearing. To avoid cycling use

- Perturbation method: perturb right hand sides to $b_i + \epsilon^o,\ i = 1, \ldots, m$, where $0 \ll \epsilon^m \leq \ldots \ll \epsilon$. The perturbed problem will be non-degenerate, so cycling will not occur.
- Bland's smallest subscript rule. By choosing the entering and leaving variable to have the smallest subscript, cycling is avoided.

When there is no cycling, the maximum number of feasible solutions is $\frac{n!}{(m!(n-m)!)}$, where m is the number of constraints and n is the number of variables. This determines the maximum number of iterations.

(c) **(i)** The problem is equivilent to maximizing $\overline{z} = -5x_1 - 2x_2 - 5x_3$. Add slack variables $s_1 \geq 0,\ s_2 \geq 0$

Basic	$\overline{z}$	x_1	x_2	x_3	s_1	s_2	
s_1	0	-3	-1	-2	1	0	-4
s_2	0	-6	-3	-5	0	1	-10
	1	5	2	5	0	0	

Basic	$\overline{z}$	x_1	x_2	x_3	s_1	s_2	
s_1	0	-1	0	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{2}{3}$
x_2	0	2	1	$\frac{5}{3}$	0	$-\frac{1}{3}$	$\frac{10}{3}$
	1	1	0	$\frac{4}{3}$	1	$\frac{1}{3}$	$-\frac{22}{3}$

Basic	$\overline{z}$	x_1	x_2	x_3	s_1	s_2	
x_1	0	1	0	$\frac{1}{3}$	-1	$\frac{1}{3}$	$\frac{2}{3}$
x_2	0	0	1	1	2	-1	2
	1	0	0	$\frac{4}{3}$	1	$\frac{1}{3}$	$-\frac{22}{3}$

Thus, the optimal solution is $x_1 = \frac{2}{3},\ x_2 = 2,\ x_3 = 0,\ z = \frac{22}{3}$.

(ii) Include a slack variable $s_3 \geq 0$ in the new constraint

Basic	$\overline{z}$	x_1	x_2	x_3	s_1	s_2	s_3	
x_1	0	1	0	$\frac{1}{3}$	-1	$\frac{1}{3}$	0	$\frac{2}{3}$
x_2	0	0	1	1	2	-1	0	2
s_3	0	0	0	-1	2	-1	1	-1
s_3	0	0	0	-1	2	-1	1	-1
	1	0	0	$\frac{4}{3}$	1	$\frac{1}{3}$	0	$-\frac{22}{3}$

Basic	$\overline{z}$	x_1	x_2	x_3	s_1	s_2	s_3	
x_1	0	1	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$
x_2	0	0	1	2	0	0	-1	3
s_2	0	0	0	1	-2	1	-1	1
	1	0	0	1	$\frac{5}{3}$	0	$\frac{1}{3}$	$-\frac{23}{3}$

The new solution is $x_1 = \frac{1}{3},\ x_2 = 3,\ x_3 = 0,\ z = \frac{23}{3}$