Question

A plane passes through the points A, B and C given by the vectors $\vec{OA} = (1, 0, -1)$, $\vec{OB} = (1, 2, 3)$ and $\vec{OC} = (0, 1, 2)$.

(a) Write down the vectors $\vec{AB}$ and $\vec{AC}$;

(b) Obtain a vector $\mathbf{n}$ perpendicular to the plane P;

(c) By writing the equation of the plane in the form $(\mathbf{r} - \mathbf{r_0}) \cdot \mathbf{n} = 0$, show that the equation of the plane is $2x - 4y + 2z = 0$. Verify that the points A, B and C all satisfy this equation

Answer

(a) $\vec{AB} = (0, 2, 4) = \mathbf{m}$ $\vec{AC} = \vec{OC} - \vec{OA} = (0, 1, 2) - (1, 0, -1) = (-1, 1, 3) = \mathbf{l}$

(b) The plane P passes through A, B and C. Hence the vector $\mathbf{n}$ is perpendicular to all vectors in the plane. Hence in particular $\mathbf{n}$ is perpendicular to $\vec{AB}$ and $\vec{AC}$ (or $\mathbf{l}$ and $\mathbf{m}$). Thus $\mathbf{n} = \mathbf{l} \times \mathbf{m}$ is perpendicular to $\mathbf{l}$ and $\mathbf{m}$.

Then $\mathbf{n} =$

$$\begin{array}{cccccc} -1 & 1 & 3 & -1 & 1 & 3 \\ & \searrow & \nearrow & \searrow & \nearrow & \\ 0 & 2 & 4 & 0 & 2 & 4 \end{array} = (-2, 4, -2)$$

(c) Then the equation of the plane is $(\mathbf{r} - \mathbf{r_0}) \cdot \mathbf{n} = 0$ with $\mathbf{r_0}$ any point in the plane.

Then

$$\begin{aligned} [\mathbf{r} - (1, 0, -1)] \cdot [-2, 4, -2] &= 0 \\ \text{or} \quad [(x, y, z) - (1, 0, -1)] \cdot [-2, 4, -2] &= 0 \\ \Rightarrow -2x + 4y + 2z + 2 - 2 &= 0 \\ \Rightarrow 2x - 4y + 2z &= 0 \end{aligned}$$

Check that this is right by putting the A, B, C into the equation.

$(1, 0, -1)$	$2(1) + 2(-2) = 0$
$(1, 2, 3)$	$2(1) - 4(2) + 2(3) = 2 - 8 + 6 = 0$
$(0, 1, 2)$	$-4(1) + 2(2) = 0$

Hence the plane does indeed pass through all the points.