Partial Differentiation
Limits

Question
Given a function $f(x, y)$ and a point in its domain (a, b). Assume that the single variable functions g and h are described as

$$\begin{aligned} g(x) &= f(x, b) \\ h(y) &= f(a, y) \end{aligned}$$

If g is continuous at $x = a$ and h is continuous at $y = b$, does this mean that f is continuous at (a, b)?
Also, does continuity of f at (a, b) mean that g is continuous at a and that h is continuous at b. Justify your answers?
Answer
Let $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$
Let $a = b = 0$. If $g(x) = f(x, 0)$ and $h(y) = f(0, y)$, then $g(x) = 0\ \forall x$, and $h(y) = 0\ \forall y$.
So g and h are continuous at 0. However f is not continuous.
If $f(x, y)$ is continuous at (a, b), then $g(x) = f(x, b)$ is continuous at $x = a$ as

$$\lim_{x \to a} g(x) = \lim_{x \to a,\ y \to b} f(x, y) = f(a, b).$$

Similarly, $h(y) = f(a, y)$ is continuous at $y = b$.