QUESTION

Say, giving your reasons, whether the set G forms a group under the given operation in each of the following cases:

(i) G is the set of subsets of a set X, with the operation of union of sets.

(ii) G is the set of subsets of a set X, with the operation of intersection of sets.

(iii) G is the set of subsets of a set X, with the operation of symmetric difference, i.e., $A \triangle B = A \cup B - A \cap B$. (You nay assume that this operation is associative.)

(iv) $G = \mathbf{R}$ with the operation $x * y = min(x, y)$.

(v) $G = \mathbf{R}$ with the operation of subtraction.

(vi) The set of functions from a set X to $\mathbf{R}$ with the operation $(f + g)$ defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in X$.

(vii) The set of odd numbers under addition.

ANSWER

(i)

$$\begin{aligned} (X' \cup Y) \cup Z &= X \cup (Y \cup Z) \\ X' \cup \emptyset &= \emptyset \cup X' = X' \\ X' \cup Y = \emptyset &\rightarrow \{x | x \in X' \text{ or } x \in Y\} = \emptyset \\ &\rightarrow X = Y = \emptyset \end{aligned}$$

so unless $X' = \emptyset$, G is not a group.

(ii)

$$\begin{aligned} (X' \cap Y) \cap Z &= \{x \in X' \cap Y | x \in Z\} \\ &= \{x | x \in X', x \in Y \text{ and } x \in Z\} \\ &= \{x \in X' \cap (Y \cap Z)\} \end{aligned}$$

$X' \cap X = X'$ so there is an identity but $X' \cap Y = X = X' = Y = X$ so there are no inverses so G is not a group.

(iii) Operation is associative.

$X'\triangle Y = X' \Leftrightarrow X' \cup Y \backslash X' \cap Y = X \Leftrightarrow \{y \in Y\backslash X'\} = \{y \in X' \cap Y\}$ so Y must be a subset of X'. Hence identity$=\emptyset$.

$X\triangle X = \emptyset$ so X' has an inverse so G is a group.

(iv) $min(x, y) = x \Rightarrow y \geq x$ since $\mathbf{R}$ has no maximal element there is no identity. So G is not a group.

(v) $(x - y) = x \Rightarrow y = 0$ and 0 is the identity.

$(x - y) - z \neq (x - (y - z))$ unless $z = 0$ so not associative $\Rightarrow$ not a group.

(vi)

$$\begin{aligned} ((f+g)+k)(x) &= (f+g)(x) + k(x) \\ &= f(x) + g(x) + k(x) \\ &= (f + (g+k))(x) \end{aligned}$$

so the operation is associative

$(f + 0)(x) = f(x)$ if $0(x) = 0 \forall x$ so $+$ has an identity.

Define $(-f)(x) = -f(x) \forall x$ so $f(x) + (-f)(x) = 0 \forall x$ so inverses exist $\forall f$ so G is a group.

(vii) $3 + 5 = 8 \notin \{$ odd numbers $\}$ so this is therefore not a group.