QUESTION

State Burnside's Formula, carefully defining the terms used in the formula. Describe the rotation group of the cube, its fixed set, and describing the orbits of the faces for each rotation.

Use this to find the number of distinct ways there are to label the faces of a cube with four colours. (As usual, "distinct" means that the labellings can be distinguished up to a rotation of the cube, so you will need to consider the action of the rotation group of the cube on the set of all possible labellings.)

ANSWER

Burnside's Formula:

Let G act on a set X. For each $g \in G$ let $X_g = \{x \in X | gx = x\}$ and for each $x \in X$ let the orbit of x, denoted $G_x = \{gx | g \in G\}$. Then the number of orbits is given by

$$\frac{1}{|G|} \sum_{g \in G} |X_g|$$

G has 24 elements

type 0 e - order 1 - entire cube is fixed, each face is its own orbit.

type 1 8 rotations of order 3, each with an invariant diagonal and two orbits of faces each of length 3.

type 2 6 rotations of order 2 each fixing a line joining the midpoints of diagonally opposite edges and having 3 orbits each of length 2.

type 3 3 rotations of order 2 fixing a line joining the midpoints of opposite faces and having 2 orbits of length 2 and 2 of length 1.

type 4 6 rotations of order 4 and 2 of length 1, 1 orbit of length 4 and 2 of length 1.

Each face can have one of 4 colour so there are 4^6 colourings available. G acts on the set X of these colourings and number of orbits of colouringd=number of "distinct" ways to colour the cube.

By Burnside this is $\frac{1}{|G|} \sum_{g \in G} |X_g|$.

$\vert X_e\vert =$	$\vert X\vert =$	$4^6 =$	4096
$\vert X_g\vert =$	4^2	type 1=	16
	4^3	type 2=	64
	4^4	type 3=	256
	4^3	type 4=	64

so number of distinct colourings

$$
\begin{aligned}
&= \frac{1}{24}\left(4^6 + 8.4^2 + 6.4^3 + 3.4^4 + 6.4^3\right) \\
&= \frac{1}{24}(4096 + 128 + 384 + 768 + 384) \\
&= 240
\end{aligned}
$$