QUESTION
Define the following terms:

**(i)** homomorphism

**(ii)** isomorphism

Let $G$ be a group. Show that the function $f : G \Rightarrow G$ given by $f(g) = g^2$ is a homomorphism if and only if $G$ is abelian. Show that if $G$ is finite, abelian and has no elements of order 2 then $f$ is an isomorphism.
Define Euler's $\phi$-function, state Euler's $\phi$ formula and use it to find the number of elements $m \in \mathbf{Z}_8$ such that $\langle m \rangle = \mathbf{Z}_8$.
ANSWER

**(i)** A homomorphism is a function $f : G \to H$ between two groups $(G, e_G, *)$ and $(H, e_H, .)$ such that for any elements $g_1, g_2 \in G$, $f(g_1 * g_2) = f(g_1).f(g_2)$.

**(ii)** An isomorphism is a bijective homomorphism.

Suppose first that $G$ is abelian, so for any $g_1, g_2 \in G$, $g_1 * g_2 = g_2 * g_1$. Then $f(g_1 * g_2) = (g_1 * g_2)^2 = g_1 * g_2 * g_1 * g_2 = g_1 * (g_1 * g_2) * g_2 = g_1^2 * g_2^2 = g(g_1) * f(g_2)$ so $f$ is a homomorphism.
Now suppose $h$ is a homomorphism so for any $g_1, g_2 \in G$ $f(g_1 * g_2) = f(g_1) * f(g_2)$.
Then $(g_1 * g_2)^2 = g_1^2 g_2^2 \forall g_1, g_2 \in G$ so

$$\begin{aligned}
& g_1 * g_2 * g_1 * g_2 = g_1 * g_1 * g_2 * g_2 \\
\Rightarrow\ & g_1^{-1} * g_1 * g_2 * g_1 * g_2 * g_2^{-1} = g_1^{-1} * g_1 * g_1 * g_2 * g_2 * g_2^{-1} \\
\Rightarrow\ & g_2 * g_1 = g_1 * g_2 \forall g_1, g_2 \in G
\end{aligned}$$

If $G$ is abelian then $f$ is a homomorphism. If $G$ has no elements of order 2 then $f(g) = e \Leftrightarrow g^2 = e \Leftrightarrow g = e$ so $\ker f = \{e\}$. Hence $f$ is injective. Since $G$ is finite any injective function $f : G \to G$ is also surjective so $f$ is an isomorphism.
$\phi(n)$=number of integers less than $n$ and coprime to it.
Euler's formula $\phi(n) = \sum_{d|n} \phi(d)$
$\phi(8)$=number of generators of $Z_8$= number of integers coprime to 8 and less than 8= number of odd integers less then 8= 4 so there are 4 generators.