QUESTION

Decide for each of the following statements whether or not it is true giving a brief explanation for your answer.

(i) The odd permutations in S_n form a subgroup.

(ii) If G and G' are isomorphic groups then every subgroup of G is isomorphic to a subgroup of G'.

(iii) Every group is isomorphic to a subgroup of a permutation group.

(iv) For every positive integer n there is a non-abelian group with precisely n elements.

(v) If $f : G \to G'$ is a surjective homomorphism then the order of G' divides the order of G.

(vi) Every subgroup of an abelian group is normal.

ANSWER

(i) False

If m, n are odd, $\tau_1, \tau_2, \ldots, \tau_m, \tau_{m+1}, \ldots, \tau_{m+n}$ are transpositions and $\sigma = \tau_1 \ldots \tau_m$, $\sigma' = \tau_{m+1} \ldots \tau_{m+n}$ then $\sigma, \sigma' \in \{$ odd permutations$\}$ but $\sigma\sigma'$ is even.

(ii) True

Let $f : G \to G'$ be an isomorphism, and $H < G$. Then $f|_H : H \to f(H)$ is bijective and $\forall h_1, h_2 \in H\ f(h_1.h_2) = f(h_1).f(h_2)$ since f is an isomorphism. So $H \cong f(H)$.

(iii) True

Let S_G denote the group of permutations of G and for each $g \in G$ define $\sigma_g \in S_G$ by $\sigma_g(h) = g.h$. $f : \begin{matrix} G \Rightarrow S_G \\ g \mapsto \sigma_g \end{matrix}$ is injective by the cancellation lemma and $f(gg')(h) = \sigma_{gg'}(h) = gg'h = g(g'h) = \sigma_g(g'h) = \sigma_g \circ \sigma_{g'}(h)$ so it is a homomorphism. Hence f is an isomorphism from G to $Im(f)$.

(iv) False

If n is prime then any group of order n is cyclic and hence abelian.

(v) True

Let $K = \ker f$ so $\frac{G}{K} \cong G'$ and $|G'| = \frac{|G|}{|K|}$ i.e. $|G| = |K||G'|$

(vi) True

A subgroup $H < G$ is normal $\Leftrightarrow$ $gH = Hg$ for every $g \in G$. In an Abelian group $gh = hg \forall h, g \in G$ so $gH = Hg$.