Question
Let ℓ_1 be the hyperbolic line contained in the Euclidean line $\{z \in \mathbf{H} | \text{Re}(z) = 2\}$, and let ℓ_2 be the hyperbolic line contained in the Euclidean circle with center -3 and radius 8. Determine all the elements of Möb($\mathbf{H}$) taking ℓ_1 to ℓ_2.

Answer
ℓ_1 has endpoints at infinity $x_1 = 2, y_1 = \infty$
ℓ_2 has endpoints at infinity $x_2 = -11, y_2 = 5$

Consider $p(z) \in$Möb$^+(\mathbf{H})$. $p(z) = \dfrac{11(z-2)+5}{-(z-2)+1} = \dfrac{11z-17}{-z+3}$

Then, $p(2) = 5, p(\infty) = -11$, and $\det(p) = 16 > 0$, and so $p(z) \in$Möb$^+(\mathbf{H})$ and $p(\ell_1) = \ell_2$.

If $m \in$Möb($\mathbf{H}$), $m(\ell_1) = \ell_2$, then $p^{-1} \circ m(\ell_1) = \ell_1$, and so $m = p \circ n, n(\ell_1) = \ell_1$.

$\text{stab}_{\text{Möb}(\mathbf{H})}(\ell_1)$ is generated by:

$$\mu(z) = -\bar{z} + 4$$
$$\nu(z) = 2 + \frac{1}{(z-2)}$$
$$\lambda(z) = az + 2(1-a) \quad (a > 0)$$

(<u>no</u> parabolics)

(conjugate the generators of $\text{stab}_{\text{Möb}(\mathbf{H})}(I)$ by $z \mapsto z+2$, where I is the positive imaginary axis).

So, the set of all Möb($\mathbf{H}$) taking ℓ_1 to ℓ_2 is generated by

$$p \circ \mu(z) = \frac{11(-\bar{z}+4)-17}{\bar{z}-4+3} = \frac{-11\bar{z}+27}{\bar{z}-1}$$

$$p \circ \nu(z) = \frac{11\nu(z)-17}{-\nu(z)+3} = \frac{-6z+23}{2z-5}$$

$$p \circ \lambda(z) = \frac{11az+22(1-a)-17}{-az-2(1-a)+3} \quad (a > 0)$$