Question
For the circle A given in Problem 1, determine the general form of an element of $\text{Möb}(A) = \{m \in \text{Möb} \mid m(A) = A\}$.

Further, determine which elements of $\text{Möb}(A)$ **do not** interchange the two discs determined by A, and which **do**.

Answer
$m \in$Möb($\mathbf{R}$) has one of the following forms:

$$p(z) = \frac{az+b}{cz+d} \quad a,b,c,d \in \mathbf{R},\ \mathbf{ad} - \mathbf{bc} = \pm\mathbf{1}.$$
$$q(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \quad a,b,c,d \in \mathbf{R},\ \mathbf{ad} - \mathbf{bc} = \pm\mathbf{1}.$$

Calculate mpm^{-1} and mqm^{-1} with m as in problem 1.

mpm^{-1}

$$= \begin{pmatrix} -4-8i & 4+6i \\ -2 & 1+i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1+i & -4-6i \\ 2 & -4-8i \end{pmatrix}$$

$$= \begin{pmatrix} \begin{array}{c} (4-12i)a - (8+16i)b \\ +(-2+10i)c + (8+12i)d \end{array} & \begin{array}{c} (-32+56i)a + (-48+16i)b \\ +(-20+48i)c + (32+-56i)d \end{array} \\ & \\ \begin{array}{c} -2(1+i)a - 4b \\ +2ic + 2(1+i)d \end{array} & \begin{array}{c} (8+12i)a + (8+16i)b \\ +(2-10i)c + (4-12i)d \end{array} \end{pmatrix}$$

where $a,b,c,d \in \mathbf{R}$ and $ad - bc = \pm 1$ (mpm^{-1} does <u>not</u> interchange the two discs determined by A if and only if $ad - bc = 1$).

mqm^{-1}: <u>careful with $\bar{z}$s in the compositions</u>

mqm^{-1}

$$= \begin{pmatrix} -4-8i & 4+6i \\ -2 & 1+i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1-i & -4+6i \\ 2 & -4+8i \end{pmatrix}$$

since $m^{-1}(z)$ gets conjugated in $q(z)$

$$\begin{pmatrix} (-4-8i)a+(4+6i)c & (-4-8i)b+(4+6i)d \\ -2a+(1+i)c & -2b+(1+i)d \end{pmatrix} \begin{pmatrix} 1-i & -4+6i \\ 2 & -4+8i \end{pmatrix}$$

$$= \begin{bmatrix} \begin{matrix} (-12-4i)a-(8+16i)b \\ +(10+2i)c+(8+12i)d \end{matrix} & \begin{matrix} (64+16i)a+80b \\ -52c+(-64+8i)d \end{matrix} \\ & \\ \begin{matrix} -2(1-i)a-4b \\ +2c+2(1+i)d \end{matrix} & \begin{matrix} (8-12i)a+(8-16i)b \\ +(-10+2i)c+(-12+2i)d \end{matrix} \end{bmatrix}$$

where $a, b, c, d \in \mathbf{R}$ and $ad - bc = \pm 1$ (mqm^{-1} does not interchange the two discs determined by A if and only if $ad - bc = -1$).