Question

Let A be the Euclidean circle in $\mathbf{C}$ with center $2+i$ and radius 3. Construct two (non-equal and) explicit Möbius transformations m and n satisfying $m(\overline{\mathbf{R}}) = n(\overline{\mathbf{R}}) = A$.

Verify that $m \circ C \circ m^{-1} = n \circ C \circ n^{-1}$, where $C(z) = \overline{z}$ is complex conjugation.

Answer

$\underline{m(\bar{\mathbf{R}}) = A}$:

Choose 3 points on A: $z_1 = 5+i; z_2 = -1+i; z_3 = 2+4i$

$m^{-1}(z_1) = 0, m^{-1}(z_2) = 1, m^{-1}(z_3) = \infty$

$$\begin{aligned}
m^{-1}(z) &= \frac{z-(5+i)}{z-(2+4i)} \cdot \frac{(-1+i)-(2+4i)}{(-1+i)-(5+i)} \\
&= \frac{z-(5+i)}{z-(2+4i)} \cdot \frac{(-3-3i)}{-6+0} \\
&= \frac{z-(5+i)}{z-(2+4i)} \cdot \frac{(1+i)}{2} \\
&= \frac{(1+i)z-4-6i}{2z-4-8i} \\
m(z) &= \frac{(-4-8i)z+4+6i}{-2z+1+i}
\end{aligned}$$

Choose another three points on A: $w_1 = -1+i; w_2 = 2+4i; w_3 = 2-2i$

$n^{-1}(w_1) = 0; n^{-1}(w_2) = 1; n^{-1}(w_3) = \infty$

$$\begin{aligned}
n^{-1}(z) &= \frac{z-(-1+i)}{z-(2-2i)} \cdot \frac{(2+4i)-(2-2i)}{(2+4i)-(-1+i)} \\
&= \frac{z+1-i}{z-2+2i} \cdot \frac{(6i)}{3+3i} \\
&= \frac{z+1-i}{z-2+2i} \cdot \frac{(2i)}{1+i} \\
&= \frac{2iz+2+2i}{(1+i)z-4} \\
n(z) &= \frac{-4z-2-2i}{-(1+i)z+2i}
\end{aligned}$$

$$
\begin{aligned}
m \circ C \circ m^{-1}(z) &= m(\overline{m^{-1}(z)}) \\
&= m\left[\frac{(1-i)\bar{z}-4+6i}{2\bar{z}-4+8i}\right] \\
&= \frac{(-4-8i)[(1-i)\bar{z}-4+6i]+(4+6i)(2\bar{z}-4+8i)}{-2[(1-i)\bar{z}-4+6i]+(1+i)(2\bar{z}-4+8i)} \\
&= \frac{(-4+8i)\bar{z}+16i}{4i\bar{z}-4-8i}
\end{aligned}
$$

$$
\begin{aligned}
n \circ C \circ n^{-1}(z) &= n(\overline{n^{-1}(z)}) \\
&= n\left[\frac{-2i\bar{z}+2-2i}{[(1-i)\bar{z}-4]}\right] \\
&= \frac{-4(-2i\bar{z}+2-2i)-(2+2i)[(1-i)\bar{z}-4]}{-(1+i)(-2i\bar{z}+2-2i)+2i[(1-i)\bar{z}-4]} \\
&= \frac{(-4+8i)\bar{z}+16i}{4i\bar{z}-4-8i} = m \circ C \circ m^{-1}(z), \text{ as desired}
\end{aligned}
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