Question

An extremely well constructed rocket has mass ratio (initial to final mass) of 10. A new fuel is developed that has an exhaust velocity of 4500ms^{-1}. The fuel burns at a constant rate for 300s. Calculate the maximum velocity of this single stage rocket assuming constant acceleration due to gravity. If the escape velocity of a particle from the earth is 11.2kms^{-1}, can this rocket reach the moon?

Answer

PICTURE

By Newton's 2nd law:

$$\begin{aligned}
-m(t)g &= \text{rate of change of momentum} \\
&= m(t)\frac{dv}{dt} + u\frac{dm}{dt} \\
\frac{dv}{dt} &= -g - \frac{u}{m}\frac{dm}{dt} \\
\text{Integrating w.r.t. } t: \int dv &= \int -g\,dt - \int \frac{u}{m}\,dm \\
v &= -gt - u\ln|m| + c \\
&\quad \text{using the fact that } t=0, v=0, m=m_0 \\
\Rightarrow c &= u\ln|m_0| \\
\text{therefore } v(t) &= -gt + u\ln|m_0| - u\ln|m| \\
v(t) &= -gt + u\ln\frac{m_0}{m(t)}
\end{aligned}$$

In this case $m(t) = m_0 - \alpha t$

Where the burn rate $\alpha = \frac{0.9m_0}{r}$, and where $r = 300\text{s}$

We must first check that it lifts off.

i.e. Thrust $= u\alpha = 4.5\times 10^3 \times \frac{9.0m_0}{300} = 13.3m_0\text{ms}^{-1} \geq m_0 g$

(where $g = 9.81\text{ ms}^{-2}$). Therefore the rocket does lift off.

Now $v(0) = 0$. Are there any maxima of $v(t)$ on the range $0 < t < r$?

$\dfrac{dv}{dt} = -g + \dfrac{\alpha n}{m(t)} = g\left[\dfrac{\alpha n}{m(t)g} - 1\right] > 0$

as $\alpha u > m_0 g > m(t)g$ from the lift-off condition above.

Hence there are no internal maxima, so the maximum velocity is at $t = r$;

$v_{\max} = -g30 + u\ln 10 = 7400ms^{-1}$.

Thus the final velocity is less than escape velocity of 11.2kms^{-1} and so the rocket cannot reach the moon.