Applications of Partial Differentiation
Extremes

Question

(a) Find the critical points of the function $z = g(x, y)$ that satisfies the equation

$$e^{2zx-x^2} - 3e^{2zy+y^2} = 2.$$

(b) Classify the critical points of g

Answer

(a) Differentiate the equation

$$e^{2zx-x^2} - 3e^{2zy+y^2} = 2.$$

with respect to x and y, regarding z as a function of x and y:

$$e^{2zx-x^2}\left(2x\frac{\partial z}{\partial x} + 2z - 2x\right) - 3e^{2zy+y^2}\left(2y\frac{\partial z}{\partial x}\right) = 0 \qquad (*)$$

$$e^{2zx-x^2}\left(2x\frac{\partial z}{\partial y}\right) - 3e^{2zy+y^2}\left(2y\frac{\partial z}{\partial y} + 2z + 2y\right) = 0 \qquad (**)$$

For a critical point we have $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, and it follows from the equations above that $z = x$ and $z = -y$.

Substituting these into the given equation we get:

$$\begin{aligned} e^{z^2} - 3e^{-z^2} &= 2 \\ (e^{z^2})^2 - 2e^{z^2} - 3 &= 0 \\ (e^{z^2} - 3)(e^{z^2} + 1) &= 0 \end{aligned}$$

Thus $e^{z^2} = 3$ or $e^{z^2} = -1$.

Since $e^{z^2} = -1$ is not possible, we have $e^{z^2} = 3$, so $z = \pm\sqrt{\ln 3}$.

The critical points are $(\sqrt{\ln 3}, -\sqrt{\ln 3})$ and $(-\sqrt{\ln 3}, \sqrt{\ln 3})$.

(b) Use the second derivative test to classify the critical points found in part (a). We need to calculate the second partials

$$\begin{aligned} A &= \frac{\partial^2 z}{\partial x^2} \\ B &= \frac{\partial^2 z}{\partial x \partial y} \\ C &= \frac{\partial^2 z}{\partial y^2} \end{aligned}$$

to do this, differentiate (*) and (**) from (a).

Differentiating (*) with respect to x gives

$$\begin{aligned} e^{2zx-x^2} & \left[\left(2x\frac{\partial z}{\partial x} + 2z - 2x \right)^2 \right. \\ & \left. +4\frac{\partial z}{\partial x} + 2x\frac{\partial^2 z}{\partial x^2} - 2 \right] \\ -3e^{2zy+y^2} & \left[\left(2y\frac{\partial z}{\partial x} \right)^2 + 2y\frac{\partial^2 z}{\partial x^2} \right] = 0 \end{aligned}$$

At a critical point, $\frac{\partial z}{\partial x} = 0$, $z = x$, $z = -y$ and $z^2 = \ln 3$, so

$$3\left(2x\frac{\partial^2 z}{\partial x^2} - 2 \right) - \frac{3}{3}\left(2y\frac{\partial^2}{\partial x^2} \right) = 0$$

$$A = \frac{\partial^2 z}{\partial x^2} = \frac{6}{6x - 2y}$$

Differentiating (**) with respect to y give

$$\begin{aligned} e^{2zx-x^2} & \left[\left(2x\frac{\partial z}{\partial y} \right)^2 + 2x\frac{\partial^2 z}{\partial y^2} \right] \\ -3e^{2zy+y^2} & \left[\left(2y\frac{\partial z}{\partial y} + 2z + 2y \right)^2 \right. \\ & \left. +4\frac{\partial z}{\partial y} + 2y\frac{\partial^2 z}{\partial y^2} + 2 \right] = 0 \end{aligned}$$

and evaluation at a critical point gives

$$3\left(2x\frac{\partial^2 z}{\partial y^2} \right) - \frac{3}{3}\left(2y\frac{\partial^2 z}{\partial y^2} + 2 \right) = 0$$

$$C = \frac{\partial^2 z}{\partial y^2} = \frac{6}{6x - 2y}$$

Finally, differentiating (*) with respect to y gives

$$e^{2zx - x^2}\left[\left(2x\frac{\partial z}{\partial x} + 2z - 2x\right)\left(2x\frac{\partial z}{\partial y}\right) + 2x\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial z}{\partial y}\right]$$

$$-3e^{2zy + y^2}\left[\left(2y\frac{\partial z}{\partial y} + 2z + 2y\right)\left(2y\frac{\partial z}{\partial x}\right) + 2\frac{\partial z}{\partial x} + 2y\frac{\partial^2 z}{\partial x}\partial y\right] = 0$$

and evaluating at a critical point,

$$(6x - 2y)\frac{\partial^2}{\partial x}\partial y = 0$$

so that

$$B = \frac{\partial^2}{\partial x \partial y} = 0$$

At the critical point $(\sqrt{\ln 3}, -\sqrt{\ln 3})$ we have

$$\begin{aligned} A &= \frac{6}{8\ln 3} \\ B &= 0 \\ C &= \frac{2}{8\ln 3} \end{aligned}$$

so $B^2 - AC < 0$, and f has a local minimum at that critical point.
At $(-\sqrt{ln3}, \sqrt{\ln 3})$ we have

$$\begin{aligned} A &= -\frac{6}{8\ln 3} \\ B &= 0 \\ C &= -\frac{2}{8\ln 3} \end{aligned}$$

so $B^2 - AC < 0$ and f has a local max at that point.