Applications of Partial Differentiation
Extremes

Question

Find the three positive numbers a, b and c given that the sum of these numbers is 30 and for which the expression ab^2c^3 is maximum.

Answer

It is given that

$$a > 0,\ b > 0,\ c > 0$$

$$\text{and} \quad \mathrm{a + b + c = 30}$$

and we want to maximize the following:

$$\begin{aligned} P &= ab^2c^3 = (30 - b - c)b^2c^3 \\ &= 30b^2c^3 - b^3c^3 - b^2c^4 \end{aligned}$$

Since $P = 0$ if $b = 0$ or $c = 0$ or $b + c = 30$ (i.e. $a = 30$), the maximum value of P will occur at a critical point (b, c) satisfying $b > 0$, $c > 0$ and $b + c < 30$. For CP:

$$\begin{aligned} 0 = \frac{\partial P}{\partial b} &= 60bc^3 - 3b^2c^3 - 2bc^4 \\ &= bc^3(60 - 3b - 2c) \\ 0 = \frac{\partial P}{\partial c} &= 90b^2c^2 - 3b^3c^2 - 4b^2c^3 \\ &= b^2c^2(90 - 3b - 4c) \end{aligned}$$

Hence $9b + 6c = 180 = 6b + 8c$, from which we obtain $3b = 2c = 30$.
The three numbers are $b = 10$, $c = 15$ and $a = 30 - 10 - 15 = 5$.