Applications of Partial Differentiation
Extremes

Question
Given the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2},$$

find the volume of the largest rectangular box that can be placed inside. (The faces must be parallel to the coordinate planes.)

Answer
Let (x,y,z) be the coordinates of the corner of the box that is the first octant of space.
Thus $x,\ y,\ z\geq 0$ and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The volume of the box is

$$V=(2x)(2y)(2z)=8cxy\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

for $x\geq 0$, $y\geq 0$ and $(x^2/a^2)+(y^2/b^2)=1$, the maximum value of V^2 and hence of V, will occur at a critical point of V^2 where $x>0$ and $y>0$.
For CP:

$$\begin{aligned}0=\frac{\partial V^2}{\partial x} &= 64c^2\left(2xy^2-\frac{4x^3y^2}{a^2}-\frac{2xy^4}{b^2}\right)\\ &= 128c^2xy^2\left(1-\frac{2x^2}{a^2}-\frac{y^2}{b^2}\right)\\ 0=\frac{\partial V^2}{\partial y} &= 128c^2x^2y\left(1-\frac{x^2}{a^2}-\frac{2y^2}{b^2}\right)\end{aligned}$$

Hence we must have

$$\frac{2c^2}{a^2}+\frac{y^2}{b^2}=1=\frac{x^2}{a^2}+\frac{2y^2}{b^2}.$$

so that $x^2/a^2=y^2/b^2=1/3$ and $x=a/\sqrt{3}$, $y=b/\sqrt{3}$.
The largest box has volume

$$V=\frac{8abc}{3}\sqrt{1-\frac{1}{3}-\frac{1}{3}}=\frac{8abc}{3\sqrt{3}}\ textrmcubicunits.$$