Applications of Partial Differentiation
Extremes

Question

Find the the dimensions of an open top rectangular box. It is given that the box has volume V and the five faces give the box the least possible surface area.

Answer

Let the length, width and height of the box be x, y and z respectively. Then $V = xyz$. The total surface area of the bottom and sides is

$$\begin{aligned} S &= xy + 2xz + 2yz = xy + 2(x+y)\frac{V}{xy} \\ &= xy + \frac{2V}{x} + \frac{2V}{y} \end{aligned}$$

where $x > 0$ and $y > 0$. Since $S \to \infty$ as $x \to 0+$ or $y \to 0+$ or $x^2 + y^2\ to \infty$, S must have a minimum value at a critical point in the first quadrant.

For CP:

$$\begin{aligned} 0 = \frac{\partial S}{\partial x} &= y - \frac{2V}{x^2} \\ 0 = \frac{\partial S}{\partial y} &= x - \frac{2V}{y^2} \end{aligned}$$

Thus $x^2y = 2V = xy^2$, so that $x = y = (2V)^{1/3}$ and z $= V/(2V)^{2/3} = 2^{-2/3}V^{1/3}$.