Applications of Partial Differentiation
Extremes

Question

Find the minimum value of

$$f(x, y) = x + 8y + \frac{1}{xy}$$

in the first quadrant ($x > 0$, $y > 0$).
How do you know that a minimum exists?

Answer

$$\begin{aligned} f(x,y) &= x + 8y + \frac{1}{xy} && x > 0,\ y > 0 \\ f_1(x,y) &= 1 - \frac{1}{x^2 y} = 0 && \Rightarrow\ x^2 y = 1 \\ f_2(x,y) &= 8 - \frac{1}{xy^2} = 07 \Rightarrow\ 8xy^2 = 1 \end{aligned}$$

The critical points must satisfy

$$\frac{x}{y} = \frac{x^2 y}{xy^2} = 8$$

that is $x = 8y$. Also, $x^2 y = 1$, so $64y^3 = 1$.
Thus $y = 1/4$ and $x = 2$; the critical point is $(2, \frac{1}{4})$.
Since $f(x, y) \to \infty$ if $x \to 0+$, $y \to 0+$, or $x^2 + y^2 \to \infty$, the critical point must give a minimum value for f.
The minimum value is $f(2, \frac{1}{4}) = 2 + 2 + 2 = 6$.