Applications of Partial Differentiation
Extremes

Question

Find the minimum and maximum values of

$$f(x, y, z) = xyze^{-x^2-y^2-z^2}.$$

How do you know that extreme values exist?

Answer

$$\begin{aligned}
f(x, y, z) &= xyze^{-x^2-y^2-z^2} \\
f_1(x, y, z) &= yz(1 - 2x^2)e^{-x^2-y^2-z^2} \\
f_2(x, y, z) &= xz(1 - 2y^2)e^{-x^2-y^2-z^2} \\
f_3(x, y, z) &= xy(1 - 2z^2)e^{-x^2-y^2-z^2}
\end{aligned}$$

Any critical point must satisfy

$$\begin{aligned}
yz(1 - 2x^2) &= 0 \\
&\text{i.e. y = 0 or z = 0 or x} = \pm\frac{1}{\sqrt{2}} \\
xz(1 - 2y^2) &= 0 \\
&\text{i.e. x = 0 or z = 0 or y} = \pm\frac{1}{\sqrt{2}} \\
xy(1 - 2z^2) &= 0 \\
&\text{i.e. y = 0 or y = 0 or z} = \pm\frac{1}{\sqrt{2}}
\end{aligned}$$

Since $f(x, y, z)$ is positive at some points, negative at others, and approaches 0 as (x, y, z) recedes to infinity, f must have maximum and minimum values at critical points.
Since $f(x, y, z) = 0$ if $x = 0$ or $y = 0$ or $z = 0$, the max and min values must occur among the eight critical points where

$$\begin{aligned}
x &= \pm 1/\sqrt{2} \\
y &= \pm 1/\sqrt{2} \\
z &= \pm 1/\sqrt{2}
\end{aligned}$$

At four of these points, f has the value $\frac{1}{2\sqrt{2}}e^{-3/2}$, the maximum value.
At the other four f has the value $-\frac{1}{2\sqrt{2}}e^{-3/2}$, the minimum value.