Applications of Partial Differentiation
Extremes

Question

Find the minimum and maximum values of

$$f(x,y) = xye^{-x^2-y^2}.$$

Answer

$$\begin{aligned} f(x,y) &= xye^{-x^2-y^2} \\ f_1(x,y) &= y(1-2x^2)e^{-(x^2+y^4)} \\ f_2(x,y) &= x(1-4y^4)e^{-(x^2+y^4)} \end{aligned}$$

For critical points $y(1-2x^2)=0$ and $x(1-4y^4)=0$. The critical points are:

$$(0,0), \quad \left(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

We have

$$\begin{aligned} f(0,0) &= 0 \\ f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}e^{-3/4} > 0 \\ f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}e^{-3/4} < 0 \end{aligned}$$

Since $f(x,y) \to \infty$ as $x^2+y^2 \to \infty$, the maximum and minimum of f are

$$\frac{1}{2}e^{-3/4} \quad \text{and} \quad -\frac{1}{2}\mathrm{e}^{-3/4}$$

respectively.