Applications of Partial Differentiation
Extremes

Question
Show that there exists a local maximum point of the function

$$f(x, y, z) = 4xyz - x^4 - y^4 - z^4$$

at $(1, 1, 1)$.
Answer

$$\begin{aligned} D &= f(1+h, 1+k, 1+m) - f(1,1,1) \\ &= 4(1+h)(1+k)(1+m) - (1+h)^4 - (1+k)^4 - (1+m)^4 - 1 \\ &= 4(1+h+k+m+hk+hm+km+hkm) \\ &\quad -(1+4h+6h^2+4h^3+h^4) \\ &\quad -(1+4k+6k^2+4k^3+k^4) \\ &\quad - * 1+4m+6m^2+4m^3+m^4) - 1 \\ &= 4(hk+hm+km) - 6(h^2+k^2+m^2) + \cdots \end{aligned}$$

where $\cdots$ stands for terms of degree 3 and 4 in the variables h, k and m. Completing some squares with the quadratic terms leads us to

$$D = -2[(h-k)^2 + (k-m)^2 + (h-m)^2 + h^2 + k^2 + m^2] + \cdots$$

which is negative if $|h|$, $|k|$ and $|m|$ are small and not all 0. (This is due to the fact that the terms of degree 3 and 4 are smaller in size than the quadratic terms when the variables are small.)
Hence f has a local maximum value at $(1, 1, 1)$.