Applications of Partial Differentiation
Extremes

Question

Find and classify the critical points of the function

$$f(x,y) = x^2 y e^{-(x^2+y^2)}$$

Answer

$$\begin{aligned} f_1(x,y) &= 2xy(1-x^2)e^{-(x^2+y^2)} \\ f_2(x,y) &= x^2(1-2y^2)e^{-(x^2+y^2)} \\ A = f_{11}(x,y) &= 2y(1-5x^2+2x^4)e^{-(x^2+y^2)} \\ B = f_{12}(x,y) &= 2x(1-x^2)(1-2y^2)e^{-(x^2+y^2)} \\ C = f_{22}(x,y) &= 2x^2y(2y^2-3)e^{-(x^2+y^2)} \end{aligned}$$

For critical points

$$\begin{aligned} xy(1-x^2) &= 0 \\ x^2(1-2y^2) &= 0 \end{aligned}$$

The critical points are $(0,y)$ $\forall y$, $(\pm 1, 1/\sqrt{2})$, and $(\pm 1, -1/\sqrt{2})$.
Obviously $f(0,y) = 0$.
Also $(x,y) > 0$ if $y > 0$ and $x \neq 0$, and $f(x,y) < 0$ if $y < 0$ and $x \neq 0$.
Thus f has a local minimum at $(0,y)$ if $y > 0$, and a local maximum if $y < 0$.
the origin is a saddle point.
At $(\pm 1, 1/\sqrt{2})$: $A = C = -2\sqrt{2}e^{-3/2}$, $B = 0$, and so $AC > B^2$. Thus f has local maximum values at these two points.
At $(\pm 1, -1/\sqrt{2})$: $A = C = 2\sqrt{2}e^{-3/2}$, $B = 0$, and so $AC > B^2$. Thus f has local minimum values at these two points.
Since $f(x,y) \to 0$ as $x^2 + y^2 \to \infty$, the value

$$f(\pm 1, 1/\sqrt{2}) = e^{-3/2}/\sqrt{2}$$

is the absolute maximum value for f, and the value

$$f(\pm 1, -1/\sqrt{2}) = -e^{-3/2}/\sqrt{2}$$

is the absolute minimum value.