

Applications of Partial Differentiation
Extremes

Question

Find and classify the critical points of the function

$$f(x, y) = x^3 + y^3 - 3xy$$

Answer

$$f_1(x, y) = 3(x^2 - y)$$

$$f_2(x, y) = 3(y^2 - x)$$

For critical points: $x^2 = y$ and $y^2 = x$.

Thus $x^4 - x = 0$, that is, $x(x - 1)(x^2 + x + 1) = 0$. Thus $x = 0$ or $x = 1$.

The critical points are $(0, 0)$ and $(1, 1)$. We have

$$A = f_{11}(x, y) = 6x$$

$$B = f_{12}(x, y) = -3$$

$$C = f_{22}(x, y) = 6y$$

At $(0, 0)$: $A = C = 0$, $B = -3$. Thus $AC < B^2$, and $(0, 0)$ is a saddle point of f .

At $(1, 1)$: $A = C = 6$, $B = -3$, so $AC > B^2$. Thus f has a local minimum value at $(1, 1)$.