

Applications of Partial Differentiation
Extremes

Question

Find and classify the critical points of the function

$$f(x, y, z) = xy + x^2z - x^2 - y - z^2$$

Answer

$$\begin{aligned}f(x, y, z) &= xy + x^2z - x^2 - y - z^2 \\f_1(x, y, z) &= y + 2x(z - 1) \\f_2(x, y, z) &= x - 1 \\f_3(x, y, z) &= x^2 - 2z\end{aligned}$$

The only critical point is $(1, 1, \frac{1}{2})$. We have

$$\begin{aligned}D &= f(1+h, 1+k, 1/2+m) - f(1, 1, 1/2) \\&= 1+h+k+hk + \frac{1+2h+h^2}{2} + (1+2h+h^2)m \\&\quad -1-2h-h^2-1-k-\frac{1}{4}-m-m^2-\left(-\frac{3}{4}\right) \\&= \frac{h^2(2m-1)+2h(k+2m)-2m^2}{2}\end{aligned}$$

If $m = h$ and $k = 0$, then $D = \frac{h^2(1+2h)}{2} > 0$ for small $|h|$.

If $h = k = 0$, then $D = -m^2 < 0$ for $m \neq 0$.

Thus f has a saddle point at $(1, 1, 1/2)$.