Applications of Partial Differentiation
Extremes

Question
Find and classify the critical points of the function

$$f(x, y, z) = xyz - x^2 - y^2 - z^2$$

Answer
For critical points we have

$$\begin{aligned} 0 = f_1 &= yz - 2x \\ 0 = f_2 &= xz - 2y \\ 0 = f_3 &= xy - 2z \end{aligned}$$

Thus $xyz = 2x^2 = 2y^2 = 2z^2$, so $x^2 = y^2 = z^2$.
Hence $x^3 = \pm 2x^2$, and $x = \pm 2$ or 0. Similarly for y and z.
The only critical points are $(0, 0, 0)$, $(2, 2, 2)$, $(-2, -2, 2)$, $(-2, 2, -2)$, and $(2, -2, -2)$.
Let $\underline{u} = u\underline{i} + v\underline{j} + w\underline{k}$, where $u^2 + v^2 + w^2 = 1$. Then

$$\begin{aligned} D_u f(x, y, z) &= (yz - 2x)u + (xz - 2y)v + (xy - 2z)w \\ D_u(D_u f(x, y, z)) &= (-2u + zv + yw)u + (zu - 2v + xw)v \\ &\quad +(yu + xv - 2w)w. \end{aligned}$$

At $(0, 0, 0)$, $D_u(D_u f(0, 0, 0)) = -2u^2 - 2v^2 - 2w^2 < 0$ for $\underline{u} \neq \underline{0}$, so f has a local maximum value at $(0, 0, 0)$.
At $(2, 2, 2)$, we have

$$\begin{aligned} D_u(D_u f(2, 2, 2)) &= (-2u + 2v + 2w)u + (2u - 2v + 2w)v \\ &\quad +(2u + 2v - 2w)w \\ &= -2(u^2 + v^2 + w^2) + 4(uv + vw + wu) \\ &= -2[(u - v - w)^2 - 4vw] \end{aligned}$$

$$\begin{cases} < 0 & \text{if} \quad v = w = 0,\ u \neq 0 \\ > 0 & \text{if} \quad v = w \neq 0,\ u - v - w = 0 \end{cases}$$

Thus $(2, 2, 2)$ is a saddle point.
At $(2, -2, -2)$ we have:

$$\begin{aligned} D_u(D_u f) &= -2(u^2 + v^2 + w^2 + 2uv + 2uw - 2vw) \\ &= -2[(u + v + w)^2 - 4vw] \end{aligned}$$

$$\begin{cases} < 0 & \text{if} \quad v = w = 0,\ u \neq 0 \\ > 0 & \text{if} \quad v = w \neq 0,\ u + v + w = 0 \end{cases}$$

Thus $(2, -2, -2)$ is a saddle point. By symmetry, so are the remaining two critical points.