Applications of Partial Differentiation
Extremes

Question

Find and classify the critical points of the function

$$f(x,y) = \left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(\frac{1}{x}+\frac{1}{y}\right)$$

Answer

$$\begin{aligned}
f(x,y) &= \left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(\frac{1}{x}+\frac{1}{y}\right)\\
&= \frac{(x+1)(y+1)(x+y)}{x^2y^2}\\
f_1(x,y) &= -\frac{(y+1)(xy+x+2y)}{x^3y^2}\\
f_2(x,y) &= -\frac{(x+1)(xy+y+2x)}{x^2y^3}\\
A = f_{11}(x,y) &= \frac{2(y+1)(xy+x+3y)}{x^4y^2}\\
B = f_{12}(x,y) &= \frac{2(xy+x+y)}{x^3y^3}\\
C = f_{22}(x,y) &= \frac{2(x+1)(xy+y+3x)}{x^2y^4}
\end{aligned}$$

For critical points

$$\begin{array}{lllll} & y = & -1 & \text{or} & xy+x+2y = 0\\ \text{and} & x = & -1 & \text{or} & xy+y+2x = 0\end{array}$$

If $y=-1$, then $x=-1$ or $x-1=0$.
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If $x \neq -1$ and $y \neq -1$, then $x-y=0$, so $x^2+3x=0$.
Thus $x=0$ or $x=-3$. However, the definition of f excludes $x=0$. Thus, the only critical points are:

$$(1,-1),\ (-1,1),\ (-1,-1),\ \text{and } (-3,-3).$$

At $(1,-1)$, $(-1,1)$ and $(-1,-1)$ we have $AC=0$ and $B \neq 0$. Therefore these three points are saddle points of f.
At $(-3,-3)$, $A=C=4/243$ and $B=2/243$, so $AC>B^2$. Therefore f has a local minimum value at $(-3,-3)$.