Question

1) $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

2) $\dfrac{dy}{dx} = \dfrac{xy}{x^2+2y^2}$ $\quad(*)$

3) $\dfrac{dy}{dx} = \dfrac{x^2+xy+y^2}{x^2}$

4) $\dfrac{dy}{dx} = \dfrac{x^3+3xy^2}{3x^2y+y^3}$

5) $x\dfrac{dy}{dx} = y + x\cos^2\left(\dfrac{y}{x}\right)$

6) $\dfrac{dx}{dt} = \dfrac{x}{t} - e^{x/t}$ $\quad(*)$

Answer

(1-6 are homogeneous)

2) $\dfrac{dy}{dx} = \dfrac{xy}{x^2+2y^2} = \dfrac{\frac{y}{x}}{1+2\frac{y}{x}}$

$$\text{let } v = \frac{y}{x} \Rightarrow y = xv \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} + v = \frac{v}{1+2v^2} \Rightarrow x\frac{dv}{dx} = \frac{-2v^3}{1+2v^2} \Rightarrow \int \frac{1+2v^2}{2v^3}dv = -\int \frac{1}{x}dx$$

$$\Rightarrow \int \frac{1}{2v^3} + \frac{1}{v}dv = -\ln x + c \Rightarrow -\frac{1}{4v^2} + \ln v = -\ln x + c$$

$$-\frac{x^2}{4y^2} + \ln\left(\frac{y}{x}\right) = -\ln x + c \qquad \text{where } -\frac{x^2}{4y^2} + \ln y = c$$

4) $\dfrac{dy}{dx} = \dfrac{x^3+3xy^2}{3x^2y+y^3} = \dfrac{1+3\frac{y^2}{x^2}}{3\frac{y}{x}+\frac{y^3}{x^3}}$

$$\text{let } v = \frac{y}{x} \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} + v = \frac{1+3v^2}{3v+v^3} \Rightarrow x\frac{dv}{dx} = \frac{1-v^4}{3v+v^3} \Rightarrow \int \frac{3v+v^3}{1-v^4}dv = \int \frac{1}{x}dx$$

again use partial fractions.

$$\int \frac{v^3}{1-v^4} + \frac{3v}{2(1+v^2)} + \frac{3v}{2(1-v^2)}dv = \ln x + c$$

$$-\frac{1}{4}\ln(1-v^4) + \frac{3}{4}\ln(1+v^2) - \frac{3}{4}\ln(1-v^2) = \ln x + c$$

$$-\frac{1}{4}\ln[(1-v^2)(1+v^2)] + \frac{3}{4}\ln(1+v^2) - \frac{3}{4}\ln(1-v^2) = \ln x + c$$

$$-\frac{1}{4}\ln(1-v^2)-\frac{1}{4}\ln(1+v^2)+\frac{3}{4}\ln(1+v^2)-\frac{3}{4}\ln(1-v^2)=\ln x+c$$

$$\frac{1}{2}\ln(1+v^2)-\ln(1-v^2)=\ln x+c$$

$$\frac{\sqrt{1+v^2}}{1-v^2}=e^c x$$

$$\frac{\sqrt{1+(\frac{y}{x})^2}}{1-(\frac{y}{x})^2}=Kx$$

$$\frac{x^2+y^2}{(x^2-y^2)^2}=A$$ no simplification.

4) $\frac{dx}{dt}=\frac{x}{t}-e^{\frac{x}{t}}$ let $\frac{x}{t}=v\Rightarrow\frac{dx}{dt}=t\frac{dv}{dt}+v$

$$\Rightarrow t\frac{dv}{dt}+v=v-e^v\Rightarrow t\frac{dv}{dt}=-e^v\Rightarrow\int e^{-v}dv=\int\frac{-1}{t}dt$$

$$\Rightarrow -e^{-v}=-\ln t+c\Rightarrow v=-\ln(\ln t+c)$$

$$x=-t\ln(\ln t+c)$$