Question

Write the following initial value problems in the form of an integral equation and hence find the solution using Picard iteration.

(a) $y' = y \qquad y(0) = 1$

(b) $y' = y^2 \qquad y(0) = 1$

(c) $y' = 2x(1 + y) \qquad y(0) = 0$

Answer

(a) $y' = y \qquad y(0) = 1$ (1)

The integral equation is $y(x) - 1 = \int_0^x y(t)\, dt$ (2)

Iteration $y_{n+1}(x) = 1 + \int_0^x y(t)\, dt, \quad y_0 = 1$

Hence,

$$\begin{aligned} y_1(x) &= 1 + \int_0^x dt = 1 + x \\ y_2(x) &= 1 + \int_0^x (1+t)\, dt = 1 + x + \frac{1}{2}x^2 \\ y_3(x) &= 1 + \int_0^x (1 + x + \frac{1}{2}x^2)\, dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \end{aligned}$$

This suggests

$$y_n(x) = 1 + x + \frac{x^2}{2} + ... + \frac{x^n}{n!}$$

Prove this by induction:

True for $n = 1$.

Suppose true for n then

$$\begin{aligned} y_{n+1}(x) &= 1 + \int_0^x (1 + t + \frac{t^2}{2} + ... + \frac{t^n}{n!})\, dt \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^{n+1}}{(n+1)!} \end{aligned}$$

So true for $n \Rightarrow$ true for $n + 1$.

Since it was true for $n = 1$ then by the induction hypothesis it is true for all $n \in \mathbf{N}$.

As $n \to \infty \quad y_n(x) \to y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = e^x$

(You can check that $y = e^x$ satisfies (1))

(b) $y' = y^2 \qquad y(0) = 1$ (1)

The integral equation is $y(x) - 1 = \int_0^x y^2(t)\,dt$ (2)

Iteration $y_{n+1}(x) = 1 + \int_0^x y^2(t)\,dt, \quad y_0 = 1$

Hence,

$$\begin{aligned}
y_1(x) &= 1 + \int_0^x dt = 1 + x \\
y_2(x) &= 1 + \int_0^x (1+t)^2\,dt \\
&= 1 + x + x^2 + \frac{x^3}{3} \\
y_3(x) &= 1 + \int_0^x (1 + t + t^2 + \frac{t^3}{3})^2\,dt \\
&= 1 + \int_0^x (1 + 2t + 3t^2 \frac{8}{3}t^3 + \frac{5}{3}t^4 + \frac{2}{3}t^5 + \frac{1}{9}t^6)\,dt \\
&= 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \frac{1}{3}x^5 + \frac{1}{9}x^6 + \frac{x^7}{63}
\end{aligned}$$

this suggests

$$y_n(x) = (1 + x + x^2 + ... + x^n)p_n(x) \qquad (3)$$

where $p_n(x)$ is some polynomial.

If $y_n(x)$ has the form (3) then:

$$\begin{aligned}
y_{n+1}(x) &= 1 + \int_0^x (1 + t + t^2 + ... + t^n + t^{n+1}p_n(t))^2\,dt \\
&= 1 + \int_0^x (1 + 2t + 3t^2 + ... + nt^n + t^{n+1}q_n(t))\,dt \\
&\qquad \text{(where } q_n(t) \text{ is some polynomial)} \\
&= 1 + x + x^2 + ... + x^{n+1} + x^{n+2}r_{n+1}(x) \\
&\qquad \text{(where } r_n(x) \text{ is some polynomial)}
\end{aligned}$$

Hence it seems reasonable to suppose that as $n \to \infty$

$y_n(x) \to y(x) = 1 + x + x^2 + x^3 + ... = (1-x)^{-1}$

In fact one can prove this for $|x| < 1$ by looking more carefully at the remainder term.

One can check that $y = \dfrac{1}{1-x}$ is a solution of (1).

(c) $y' = 2x(1+y) \qquad y(0) = 0 \qquad (1)$

The integral equation is $y(x) = \int_0^x 2t(1 + y(t))\,dt \qquad (2)$

Iteration:

$$
\begin{aligned}
y_{n+1}(x) &= \int_0^x t(1 + y_n(t))\,dt \qquad y_0 = 0 \\
y_1(x) &= \int_0^x 2t\,dt = x^2 \\
y_2(x) &= \int_0^x 2t(1+t^2)\,dt \\
&= x^2 + \frac{x^4}{2} \\
y_3(x) &= \int_0^x 2t(1 + t^2 + \frac{t^4}{2})\,dt \\
&= x^2 + \frac{x^4}{2} + \frac{x^6}{6}
\end{aligned}
$$

This suggests

$$
y_n(x) = x^2 + \frac{x^4}{2!} + ... + \frac{x^{2n}}{n!} \qquad (3)
$$

If we assume this is true then

$$
\begin{aligned}
y_{n+1}(x) &= \int_0^x 2t(1 + t^2 + \frac{t^4}{2!} + ... + \frac{t^{2n}}{n!})\,dt \\
&= (2t + 2t^3 + t^5 + ... + \frac{2t^{2n+1}}{(n+1)!})\,dt \\
&= x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + ... + \frac{x^{2(n+1)}}{(n+1)!}
\end{aligned}
$$

Hence it is true for $n \Rightarrow$ true for $n+1$.

Also true for $n = 1$ so by induction $y_n(x) = x^2 + \dfrac{x^4}{2!} + ... + \dfrac{x^{2n}}{n!}$

As $n \to \infty \quad y_n(x) \to y(x) = x^2 + \dfrac{x^4}{2!} + ... = e^{x^2} - 1$

and one can check that $y = e^{x^2} - 1$ is a solution of (1)