Question

Find the Green's function for the following boundary value problem

$$y'' + y = f(x); \quad y(0) = 0, \quad y(b) = 0$$

What happens in the case $b = n\pi$ where n is an integer?
Use the Green's function to obtain the solution to

$$y'' + y = x; \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0$$

Answer

$$y'' + y = 0 \quad y(0) = 0; \quad y(b) = 0.$$

Solution is $y = A\sin x + B\cos x$

We want solutions $\begin{matrix} y_1(x) \\ y_2(x) \end{matrix}$ which vanishes when $\begin{matrix} x = b \\ x = 0 \end{matrix}$

y_2 is given by $y_2(x) = \sin x$

For y_1 we choose A and B so that

$$\begin{aligned} 0 &= A\sin b + B\cos b \\ \text{Let} \quad A = \cos b \quad \text{and} \quad & B = -\sin b \\ \text{Then} \quad y_2(x) &= \sin x\cos b - \cos x\sin b \\ &= \sin(x-b) \end{aligned}$$

$$\begin{aligned} \text{so} \quad y_1 &= \sin(x-b) \\ y_2 &= \sin x \end{aligned}$$

$$\begin{aligned} W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} \sin(x-b) & \sin x \\ \cos(x-b) & \cos x \end{vmatrix} \\ &= \sin(x-b)\cos x - \cos(x-b)\sin x \\ &= \sin(x-b-x) \\ &= \sin(-b) \\ &= -\sin b \end{aligned}$$

$$\text{Now } \tilde{G}(x,s) = \begin{cases} \dfrac{y_1(s)y_2(x)}{W(s)} & 0 \le x \le s \le b \\ \dfrac{y_1(x)y_2(s)}{W(s)} & 0 \le s \le x \le b \end{cases}$$

So in this case: $\tilde{G}(x,s) = \begin{cases} \dfrac{\sin(s-b)\sin(x)}{-\sin(b)} & 0 \le x \le s \le b \\ \dfrac{\sin(x-b)\sin(s)}{-\sin(b)} & 0 \le s \le x \le b \end{cases}$

If $b = n\pi$ then $\sin b = 0$ and $\tilde{G}(x,s)$ is not well defined by the above.
This is because there is no non-trivial solution to the boundary value problem $y'' + y = 0$ subject to $y(0) = 0$ and $y(n\pi) = 0$
If $b = \frac{\pi}{2}$ then $\sin(\frac{\pi}{2}) = 1$ and $\sin(x - \frac{\pi}{2}) = -\cos x$

$$\begin{aligned} \text{So} \quad \tilde{G}(x,s) &= \begin{cases} \cos s \sin(x) & x \le s \\ \cos x \sin s & s \le x \end{cases} \\ \text{so that} \ \ y(x) &= \int_0^x \cos x \sin s \cdot s\, ds + \int_x^{\frac{\pi}{2}} (\sin x)(\cos s) s\, ds \end{aligned}$$

Now we integrate by parts:

$$\begin{aligned} \int_0^x (\sin s) s\, ds &= [-\cos s \cdot s]_0^x + \int_0^x \cos s\, ds \\ &= -x\cos x + \sin x \\ \int_x^{\frac{\pi}{2}} (\cos s) s\, ds &= [\sin \cdot s]_x^{\frac{\pi}{2}} - \int_x^{\frac{\pi}{2}} \sin s\, ds \\ &= \frac{\pi}{2} - x\sin x - \cos x \end{aligned}$$

$$\begin{aligned} y &= -x\cos^2 x + \sin x \cos x + \frac{\pi}{2}\sin x - x\sin^2 x - \cos x \sin x \\ &= -x + \frac{\pi}{2}\sin x \end{aligned}$$

NOTE: there is a sign error in this answer, but not sure where.