Question
Find the Green's function for the following equations and use it to solve the corresponding initial-value problems

(a) $y'' + 2y' - 8y = 2e^{3x}; \quad y(0) = 1, \quad y'(0) = -\dfrac{2}{7}$

(b) $y'' + 2y' + y = e^{2x}; \quad y(0) = 0, \quad y'(0) = 0$

(c) $xy'' - 3y' = 4x - 6; \quad y(1) = 0, \quad y'(1) = 1$

Answer

(a) $y'' + 2y' - 8y = 2e^{3x}; \quad y(0) = 1, \quad y'(0) = -\dfrac{2}{7}$

The homogeneous equation is:

$$\begin{aligned} y'' + 2y' - 8y &= 0 \\ \text{A.E.} \quad m^2 + 2m - 8 &= 0 \\ (m+4)(m-2) &= 0 \\ y_1 = e^{-4x} & \qquad y_2 = e^{2x} \end{aligned}$$

Looking at the Wronskian:

$$W = \begin{vmatrix} e^{-4x} & e^{2x} \\ -4e^{-4x} & 2e^{2x} \end{vmatrix} = 6e^{-2x}$$

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} = \frac{1}{6}\left(e^{2(x-t)} - e^{-4(x-t)}\right)$$

Solution of $y'' + 2y' - 8y = 0$ satisfying $y(0) = 1$, $y'(0) = -\frac{2}{7}$

$$\begin{array}{rlrl} y(x) = Ae^{-4x} + Be^{2x} & y(0) = 1 & \Rightarrow & A + B = 1 \\ y'(x) = -4Ae^{-4x} + 2Be^{2x} & y'(0) = -\frac{2}{7} & \Rightarrow & -4A + 2B = -\frac{2}{7} \end{array}$$

Thus $A = \dfrac{8}{21}; \quad B = \dfrac{13}{21}$

So the solution to $y'' + 2y' - 8y = 2e^{3x}$ is:

$$\begin{aligned} y &= \int_0^x G(x,t)R(t)dt + \frac{8}{21}e^{-4x} + \frac{13}{21}e^{2x} \\ &= \frac{1}{3}\int_0^x \left(e^{2x}e^{t} - e^{-4x}e^{7t}\right) dt + \frac{8}{21}e^{-4x} + \frac{13}{21}e^{2x} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\left[e^{2x}e^{t} - \frac{1}{7}e^{3x}e^{7t}\right]_0^x + \frac{8}{21}e^{-4x} + \frac{13}{21}e^{2x} \\
&= \frac{1}{3}\left(e^{3x} - e^{2x} - \frac{1}{7}e^{3x} + \frac{1}{7}e^{-4x}\right)\frac{8}{21}e^{-4x} + \frac{13}{21}e^{2x} \\
&= \frac{2}{7}e^{3x} + \frac{2}{7}e^{2x} + \frac{3}{7}e^{-4x}
\end{aligned}
$$

(b) $y'' + 2y' + y = e^{2x}; \quad y(0) = 0, \quad y'(0) = 0$

The homogeneous equation is:

$$
\begin{aligned}
y'' + 2y' + y &= 0 \\
\text{A.E.}\ \ m^2 + 2m + 1 &= 0 \\
(m+1)^2 &= 0 \\
m = -1\ \ \text{twice} & \\
y_1 = e^{-x} & \qquad y_2 = xe^{-x}
\end{aligned}
$$

Looking at the Wronskian:

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-2x}$$

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} = (x-t)e^{-(x-t)}$$

Solution of $y'' + 2y' + y = 0$ satisfying $y(0) = 0$, $y'(0) = 0$ is $y(x) = 0$ (for all x)

Solution of $y'' + 2y' + y = e^{2x}$ is:

$$
\begin{aligned}
y(x) &= \int_0^x G(x,t)R(t)\,dt \\
&= \int_0^x (xe^{-x}e^{3t} - te^{-x}e^{3t})\,dt \\
&= \left[\frac{1}{3}xe^{-x}e^{3t}\right]_0^x - \left[\frac{1}{3}te^{-x}e^{3t}\right]_0^x + \frac{1}{3}\int_0^x e^{-x}e^{3t}\,dt \\
&= \left[\frac{1}{3}xe^{-x}e^{3t} - \frac{1}{3}te^{-x}e^{3t} + \frac{1}{9}e^{-x}e^{3t}\right]_0^x \\
&= \frac{1}{3}xe^{2x} - \frac{1}{3}xe^{2x} + \frac{1}{9}e^{2x} - \frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x} \\
&= \frac{1}{9}(e^{2x} - (1+3x)e^{-x})
\end{aligned}
$$

(c) $xy'' - 3y' = 4x - 6; \quad y(1) = 0, \quad y'(1) = 1$

The homogeneous equation is:

$$\begin{aligned}
xy'' - 3y' &= 0 \\
\Rightarrow x^2y'' - 3xy' &= 0 \\
\text{so Euler type equation} & \\
\text{Let } y = x^K. & \\
K(K-1) - 3K &= 0 \\
\Rightarrow K^2 - 4K &= 0 \\
K(K-4) &= 0 \\
K = 0 \quad & \text{or} \quad K = 4 \\
\text{so} \quad y_1 = x^4 \quad & \quad\quad y_2 = 1
\end{aligned}$$

Looking at the Wronskian:

$$W = \begin{vmatrix} x^4 & 1 \\ 4x^3 & 0 \end{vmatrix} = -4x^3$$

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} = \left(\frac{t^4 - x^4}{-4t^3}\right) = \frac{1}{4}\left(\frac{x^4}{t^3} - t\right)$$

Solution of $y'' - \frac{3}{x}y' = 0$ satisfying $y(1) = 0$, $y'(1) = 1$ is

$$\begin{array}{lll}
y(x) = A + Bx^4 & y(1) = 0 \;\Rightarrow & A + B = 0 \\
y'(x) = 4Bx^3 & y'(1) = 1 \;\Rightarrow & B = \frac{1}{4} \Rightarrow A = -\frac{1}{4}
\end{array}$$

Solution of $y'' - \frac{3}{x}y' = 4 - \frac{6}{x}$ is: (N.B. coefficient of y'' is 1)

$$\begin{aligned}
y &= \int_1^x G(x,t)R(t)\,dt - \frac{1}{4} + \frac{1}{4}x^4 \\
&= \frac{1}{4}\int_1^x \left(\frac{x^4}{t^3} - t\right)\left(4 - \frac{6}{t}\right)\,dt - \frac{1}{4} + \frac{1}{4}x^4 \\
&= \frac{1}{4}\int_1^x \left(\frac{4x^4}{t^3} - 4t - \frac{6x^4}{t^4} + 6\right)\,dt - \frac{1}{4} + \frac{1}{4}x^4 \\
&= \frac{1}{4}\left[-\frac{2x^4}{t^2} - 2t^2 + \frac{2x^4}{t^3} + 6t\right]_1^x - \frac{1}{4} + \frac{1}{4}x^4 \\
&= -x^2 + 2x + \frac{1}{4}x^4 - \frac{5}{4}
\end{aligned}$$