Question
Find the eigenvalues and eigenvectors of the following matrices

(i) $A = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}$

(ii) $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(iii) $C = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

Answer

(i) $A = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}$

$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow (A - \lambda)x = 0 \Rightarrow det(A - \lambda I_2) = 0$

i.e., $\begin{vmatrix} 3-\lambda & 0 \\ 0 & -4-\lambda \end{vmatrix} = -(3-\lambda)(4+\lambda) = 0$

Therefore $\underline{\lambda = 3 \text{ or } -4}$ Eigenvalues.

<u>Eigenvectors:</u>

$\underline{\lambda = 3}$

$$\begin{aligned} \begin{pmatrix} 3-3 & 0 \\ 0 & -4-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \quad 0x + 0y &= 0 \\ 0x - 7y &= 0 \\ \Rightarrow \quad y &= 0 \end{aligned}$$

Hence if $y = 0$ and $0x = 0$, x can be anything, say, k, so eigenvector is $\begin{pmatrix} k \\ 0 \end{pmatrix}$ for eigenvalue $=3$.

$\underline{\lambda = -4}$

$$\begin{pmatrix} 3-(-4) & 0 \\ 0 & -4-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 7 & 0 \\ 0 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} 7x = 0 \\ -8y = 0 \end{matrix}\right\} \Rightarrow x = y = 0$$

so eigenvector is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for eigenvalues=-4.

(ii) $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Eigenvalues and eigenvectors determined from $B\mathbf{x} = \lambda\mathbf{x}$

$$\begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvalues:

$$det\begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix} = 0$$
$$\Rightarrow \quad (1-\lambda)(1-\lambda) = 0$$
$$\Rightarrow \quad \lambda = +1 twice$$

Eigenvectors:

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} 2x = 0 \Rightarrow x = 0 \\ 0y = 0 \Rightarrow y \end{matrix} \quad \text{where y can be anything}$$

Therefore <u>eigenvectors</u> are $\begin{pmatrix} 0 \\ k \end{pmatrix}$

(iii) $C = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

Eigenvalues and eigenvectors determined from $C\mathbf{x} = \lambda\mathbf{x}$

$$\begin{pmatrix} 2-\lambda & 4 \\ 3 & 6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} & det \begin{pmatrix} 2-\lambda & 4 \\ 3 & 6-\lambda \end{pmatrix} = 0 \\ \Rightarrow \; & (2-\lambda)(6-\lambda) - 12 = 0 \\ & \lambda^2 + 12 - 6\lambda - 2\lambda - 12 = 0 \\ \Rightarrow \; & \lambda^2 = 8\lambda \end{aligned}$$

$$\Rightarrow \underline{\lambda = 0 \text{ or } 8}$$

So eigenvectors are:

$\underline{\lambda = 0}$

$$\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 2x + 4y = 0 \Rightarrow x = -2y \\ 3x + 6y = 0 \Rightarrow x = -2y \end{array}$$

so $\begin{pmatrix} -2\lambda \\ \lambda \end{pmatrix}$ is the eigenvector for $\lambda = 0$.

$\underline{\lambda = 8}$

$$\begin{pmatrix} -6 & 4 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} -6x + 4y = 0 \Rightarrow y = \dfrac{3x}{2} \\ 3x - 2y = 0 \Rightarrow y = \dfrac{3x}{2} \end{array}$$

so $\begin{pmatrix} \lambda \\ \frac{3\lambda}{2} \end{pmatrix}$ is the eigenvector for $\lambda = 8$.