Question
Find the value of k for which the equations

$$\begin{aligned} x + 2y &= 0 \\ 3x + ky - z &= 0 \\ 2x + 5y - 2z &= 0 \end{aligned}$$

have a solution, other than $x = y = z = 0$. Find the solution set for this value of k.

Answer

$$\begin{aligned} x + 2y &= 0 \quad (1) \\ 3x + ky - z &= 0 \quad (2) \\ 2x + 5y - 2z &= 0 \quad (3) \end{aligned}$$

Obviously $x = 0$ $y = 0$ $z = 0$ are solutions. Easiest way to do this is to systematically eliminate; since (1) is a "nice" equation:
(1) $\Rightarrow x = -2y \downarrow$
$\Rightarrow$ (2) becomes $3(-2y) + ky - z = 0$
(3) becomes $2(-2y) + 5y - 2z = 0$
Hence

$$\begin{aligned} (k-6)y - z &= 0 \quad (4) \\ y - 2z &= 0 \quad (5) \end{aligned}$$

(5) $\Rightarrow y = 2z$
Therefore $(k-6)(2z) - z = 0$
$\Rightarrow (2k - 13)z = 0 \quad (6)$
Now from (6) we could have $z = 0 \Rightarrow y = 0 \Rightarrow x = 0$. This isn't what we want. Another way to satisfy (60 is to have $k = \frac{13}{2}$. In this case z could be anything, say $z = \lambda$. Hence from (5), $y = 2\lambda$ and from (1), $x = -4\lambda$. Hence the solution is

$$x = -4\lambda, y = 2\lambda, z = \lambda$$

or $-\frac{x}{4} = \frac{y}{2} = z$, the equation of a line in 3-D.