Question
Solve the equations

$$\begin{aligned} 2x + 4y + 5z &= -3 \\ 4x - y - 7z &= 6 \\ 6x + 3y - z &= 3 \end{aligned}$$

by Gaussian elimination.

Answer
Augmented matrix is

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \left(\begin{array}{ccc|c} 2 & 4 & 5 & -3 \\ 4 & -1 & -7 & 6 \\ 6 & 3 & -1 & 3 \end{array} \right)$$

Want to get this in the upper echelon form

$$\left(\begin{array}{ccc|c} a & . & . & . \\ 0 & b & . & . \\ 0 & 0 & c & . \end{array} \right)$$

So to get zeros in r_3 use elementary row operations: $r_3 - (r_2 + r_1)$

$$\begin{matrix} \overline{r_1} \\ \overline{r_2} \\ \overline{r_3} \end{matrix} \left(\begin{array}{ccc|c} 2 & 4 & 5 & -3 \\ 4 & -1 & -7 & 6 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

To get zeros in $\overline{r_2}$, use elementary row operation $\overline{r_2} - 2\overline{r_1}$

$$\begin{matrix} \overline{\overline{r_1}} \\ \overline{\overline{r_2}} \\ \overline{\overline{r_3}} \end{matrix} \left(\begin{array}{ccc|c} 2 & 4 & 5 & -3 \\ 0 & -9 & -17 & 12 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

This is upper triangular form. We could divide $\overline{\overline{r_1}}$ by 2 and $\overline{\overline{r_2}}$ by -9 to get 1's, but it's not necessary to find solution.
Now use <u>back</u> substitution;

$$\begin{pmatrix} 2 & 4 & 5 \\ 0 & -9 & -17 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \\ 0 \end{pmatrix}$$

$\overline{\overline{r_3}} \Rightarrow z = 0$
hence in $\overline{\overline{r_2}}$
$-9y - 17z = 12 \Rightarrow -9y = 12 \Rightarrow y = -\dfrac{4}{3}$

hence in $\overline{\overline{r_1}}$

$$\begin{aligned} 2x + 4y + 5z = -3 \quad &\Rightarrow \quad 2x + 4\left(-\frac{4}{3}\right) + 5(0) = -3 \\ &\Rightarrow \quad 2x = \frac{16}{3} - 3 = \frac{7}{3} \\ &\Rightarrow \quad x = \frac{7}{6} \end{aligned}$$

So the solution is:

$$x = \frac{7}{6},\ y = -\frac{4}{3},\ z = 0$$