Question
Solve the equations

$$\begin{aligned} 3x + 2y - z &= 4 \\ 2x - 5y + 2z &= 1 \\ 5x + 16y - 7z &= 10 \end{aligned}$$

by Gaussian elimination.

Answer
Augmented matrix is

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 2 & -5 & 2 & 1 \\ 5 & 16 & -7 & 10 \end{array} \right)$$

Want to get this in the upper echelon form

$$\left(\begin{array}{ccc|c} a & . & . & . \\ 0 & b & . & . \\ 0 & 0 & c & . \end{array} \right)$$

Carry out elementary row operations: $\overline{r_3} = r_3 - (r_1 + r_2)$

$$\begin{matrix} \overline{r_1} \\ \overline{r_2} \\ \overline{r_3} \end{matrix} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 2 & -5 & 2 & 1 \\ 0 & 19 & -8 & 5 \end{array} \right)$$

$\overline{\overline{r_2}} = 3\overline{r_2}$

$$\begin{matrix} \overline{\overline{r_1}} \\ \overline{\overline{r_2}} \\ \overline{\overline{r_3}} \end{matrix} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 6 & -15 & 6 & 3 \\ 0 & 19 & -8 & 5 \end{array} \right)$$

$\overline{\overline{\overline{r_2}}} = \overline{\overline{r_2}} - 2\overline{\overline{r_1}}$

$$\begin{matrix} \overline{\overline{\overline{r_1}}} \\ \overline{\overline{\overline{r_2}}} \\ \overline{\overline{\overline{r_3}}} \end{matrix} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 0 & -19 & 8 & -5 \\ 0 & 19 & -8 & 5 \end{array} \right)$$

$r_3^{(iv)} = \overline{\overline{\overline{r_2}}} + \overline{\overline{\overline{r_3}}}$

$$\begin{matrix} {r_1}^{(iv)} \\ {r_2}^{(iv)} \\ {r_3}^{(iv)} \end{matrix} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 0 & -19 & 8 & -5 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

We are in trouble! The third row has vanished, and hence the third equation is a linear combination of the other two. It is linearly dependent and thus gives no further information. (This could have been spotted originally by considering

$$det \left(\begin{array}{ccc} 3 & 2 & -1 \\ 2 & -5 & 2 \\ 5 & 16 & -7 \end{array} \right) = 9 + 48 - 57 = 0.$$

Hence no inverse exists)
We thus have a rank 2 system (only 2 equations in 3 unknowns).
To solve the system we assign z arbitrarily, say $z = \lambda$ and then solve

$$\begin{aligned} 3x + 2y - \lambda &= 4 \quad (1) \\ -19y + 8\lambda &= -5 \end{aligned}$$

$(2) \Rightarrow y = \dfrac{8\lambda + 5}{19}$

Therefore in (1) $3x + \dfrac{16\lambda}{19} + \dfrac{10}{19} - \lambda = 4$

$$x = \frac{(22 + \lambda)}{19}$$

<u>or</u> $19x - 22 = \dfrac{19y - 5}{8} = z$ a line in 3-D. Here we have case 3.