Question
Find the inverses of the following matrices and verify that they are correct.

(i) $A = \begin{pmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$

(ii) $B = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix}$

(iii) $C = \begin{pmatrix} 3 & 2 & 6 \\ 5 & 2 & 11 \\ 7 & 4 & 16 \end{pmatrix}$

Answer

(i) $A = \begin{pmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

step (i)

cofactor of $A_{11} = + \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = +1$

cofactor of $A_{12} = - \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = +2$

cofactor of $A_{13} = + \begin{vmatrix} -4 & -1 \\ 2 & 1 \end{vmatrix} = -2$

cofactor of $A_{21} = - \begin{vmatrix} -2 & -1 \\ 0 & 1 \end{vmatrix} = +2$

cofactor of $A_{22} = + \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = +5$

cofactor of $A_{23} = - \begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix} = -4$

cofactor of $A_{31} = + \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} = +3$

cofactor of $A_{32} = - \begin{vmatrix} 3 & -1 \\ -4 & -1 \end{vmatrix} = +7$

cofactor of $A_{33} = +\begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} = -5$

Matrix of cofactors is thus:

$$\begin{pmatrix} +1 & +2 & -2 \\ +2 & +5 & -4 \\ +3 & +7 & -5 \end{pmatrix}$$

step (ii)

Transpose to get $adj A$

$$adj\ A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{pmatrix}$$

step (iii)

$$\begin{aligned} det\ A &= \begin{vmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - (-2)\begin{vmatrix} -4 & -1 \\ 2 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} -4 & 1 \\ 2 & 0 \end{vmatrix} \\ &= 3 - 4 + 2 = 1 \end{aligned}$$

step (iv)

$$A^{-1} = \frac{adj A}{det A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{pmatrix}$$

Check (for $A^{-1}A$ only)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{pmatrix}\begin{pmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \surd$$

(ii) $B = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$

step (i)

cofactor of $B_{11} = + \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3$

cofactor of $B_{12} = - \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -2$

cofactor of $B_{13} = + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = +2$

cofactor of $B_{21} = - \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = +4$

cofactor of $B_{22} = + \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = +1$

cofactor of $B_{23} = - \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1$

cofactor of $B_{31} = + \begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} = -7$

cofactor of $B_{32} = - \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = +2$

cofactor of $B_{33} = + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = +3$

Matrix of cofactors is thus:

$$\begin{pmatrix} -3 & -2 & 2 \\ +4 & +1 & -1 \\ -7 & +2 & +3 \end{pmatrix}$$

step (ii)

Transpose to get $adjB$

$$adj\ B = \begin{pmatrix} -3 & 4 & -7 \\ -2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

step (iii)

$$\begin{aligned} det\ B &= \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 1 \times \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \\ &= -3 + 2 + 6 = 5 \end{aligned}$$

step (iv)

$$B^{-1} = \frac{adjB}{detB} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} & -\frac{7}{5} \\ -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$$

Check (for $B^{-1}B$ only)

$$\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} & -\frac{7}{5} \\ -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \ \surd$$

(iii) $C = \begin{pmatrix} 3 & 2 & 6 \\ 5 & 3 & 11 \\ 7 & 4 & 16 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$

step (i)

cofactor of $C_{11} = + \begin{vmatrix} 3 & 11 \\ 4 & 16 \end{vmatrix} = +4$

cofactor of $C_{12} = - \begin{vmatrix} 5 & 11 \\ 7 & 16 \end{vmatrix} = -3$

cofactor of $C_{13} = + \begin{vmatrix} 5 & 3 \\ 7 & 4 \end{vmatrix} = -1$

cofactor of $C_{21} = - \begin{vmatrix} 2 & 6 \\ 4 & 16 \end{vmatrix} = -8$

cofactor of $C_{22} = + \begin{vmatrix} 3 & 6 \\ 7 & 16 \end{vmatrix} = +6$

cofactor of $C_{23} = - \begin{vmatrix} 3 & 2 \\ 7 & 4 \end{vmatrix} = +2$

cofactor of $C_{31} = + \begin{vmatrix} 2 & 6 \\ 3 & 11 \end{vmatrix} = +4$

cofactor of $C_{32} = - \begin{vmatrix} 3 & 6 \\ 5 & 11 \end{vmatrix} = -3$

cofactor of $C_{33} = + \begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} = -1$

Matrix of cofactors is thus:

$$\begin{pmatrix} 4 & -3 & -1 \\ -8 & 6 & 2 \\ 4 & -3 & -1 \end{pmatrix}$$

step (ii)

Transpose to get $adjC$

$$adj\ C = \begin{pmatrix} 4 & -8 & 4 \\ -3 & 6 & -3 \\ -1 & 2 & -1 \end{pmatrix}$$

step (iii)

$$\begin{aligned} det\ C &= \begin{vmatrix} 4 & -8 & 4 \\ -3 & 6 & -3 \\ -1 & 2 & -1 \end{vmatrix} \\ &= 4 \times \begin{vmatrix} 6 & -3 \\ 2 & -1 \end{vmatrix} - (-8) \begin{vmatrix} -3 & -3 \\ -1 & -1 \end{vmatrix} + 4 \times \begin{vmatrix} -3 & 6 \\ -1 & 2 \end{vmatrix} \\ &= 0 - 0 + 0 = 0 \end{aligned}$$

$detC = 0$. Therefore there is <u>no</u> inverse matrix such that $C^{-1}C = I_3$.