Question

Prove that each of the following statements holds in a field F, using only the axioms of a field.

1. $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$ for all a, $b \in F$;

2. $(-a) \cdot (-b) = a \cdot b$ for all a, $b \in F$;

3. $(-1) \cdot a = -a$ for all $a \in F$;

4. $(-1) \cdot (-1) = 1$.

Answer

1. Since F is a commutative group under addition, $a + (-a) = 0$. Multiplying on the right by b and applying the above fact that $0 \cdot b = 0$, we get $(a+(-a)) \cdot b = 0$. Apply the distributive law to get $a \cdot b + (-a) \cdot b = 0$. Adding the additive inverse $-(a \cdot b)$ of $a \cdot b$ to both sides and using the two facts that $-(a \cdot b) + a \cdot b = 0$ and that 0 is the additive identity, we obtain $(-a) \cdot b = -(a \cdot b)$. (Similarly, starting with $b + (-b) = 0$ and multiplying on the left by a, we get that $a \cdot (-b) = -(a \cdot b)$.) (And as above, since both $(-a) \cdot b$ and $a \cdot (-b)$ are equal to $-(a \cdot b)$, they are equal to each other.)

2. Start with $a+(-a) = 0$, and multiply both sides on the right by $b+(-b)$. Expanding out, we get $a \cdot b + a \cdot (-b) + (-a) \cdot b + (-a) \cdot (-b) = 0$. Since $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$, this becomes $a \cdot b + (-(a \cdot b)) + (-(a \cdot b)) + (-a) \cdot (-b) = 0$. Since $-(a \cdot b)$ is the additive inverse for $a \cdot b$, this becomes $-(a \cdot b) + (-a) \cdot (-b) = 0$. Adding $a \cdot b$ to both sides and simplifying, this becomes $(-a) \cdot (-b) = a \cdot b$, as desired.

3. Start with $1 + (-1) = 0$, and multiply on the right by a. Since $0 \cdot a = 0$, this becomes $(1 + (-1)) \cdot a = 0$. Expanding out, this becomes $1 \cdot a + (-1) \cdot a = 0$. Since 1 is the multiplicative identity, this becomes $a+(-1) \cdot a = 0$. Adding $-a$ to both sides and simplifying, this becomes $(-1) \cdot a = -a$, as desired.

4. Since we know already that $(-a) \cdot (-b) = a \cdot b$, we can take $a = 1$ and $b = 1$ to get $(-1) \cdot (-1) = 1 \cdot 1 = 1$, with this last equality following from the fact that 1 is the multiplicative identity.