Question

Consider the EUROPEAN AVERAGE RATE option with expiry T whose value is governed by the partial differential equation

$$V_t + (\log S)V_I + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$

where the independent variable I is defined by

$$I = \int_0^t \log S(\tau)\, d\tau.$$

(a) By considering definition of I, and the quantity

$$Q = \left(\sum_{i=1}^{n} S(t_i) \right)^{\frac{1}{n}}$$

in the limit $n \to \infty$ or otherwise, explain what kind of average is being used in the average rate option.

(b) If the payoff of the option is a function of I only, show that solutions of the form

$$\begin{aligned} V &= F(\theta, t) \\ \theta &= \frac{I + (T-t)\log S}{T} \end{aligned}$$

exist provided F satisfies

$$F_t + a(t)F_{\theta\theta} + b(t)F_\theta - rF = 0 \quad \longrightarrow \quad (1)$$

where $a(t)$ and $b(t)$ are functions that should be determined.

(c) Explain briefly why (1) is easier to solve than the original problem. If the payoff of the option depends on S as well as I, does this method still work?

Answer
We have

$$V_t + (\log S)V_I + \frac{1}{2}\sigma^2 S62V_{SS} + rSV_S - rV = 0$$

$$I = \int_0^t \log S(\tau)\, d\tau$$

(a) Consider $Q = \left(\prod_{i=1}^{n} S(_i)\right)^{\frac{1}{n}}$.

Now

$$Q = \frac{1}{n}\log\left(\prod_{i=1}^{n} S(_i)\right)$$

But since $\log(a_1 a_2 \cdots a_n) = \sum \log a_i$

$$\Rightarrow \log Q = \frac{1}{n}\sum_{i=1}^{n}\log(S(t_i)).$$

Now the <u>total</u> time of the option is finite so say

$$\begin{aligned} ndt &= T = t_n \\ \Rightarrow \log Q &= \frac{1}{T}\sum_{i=1}^{n}\log S(ndt)dt \end{aligned}$$

In the limit as $n \to \infty$ we therefore find that

$$\log Q \to \frac{1}{T}\int_0^T \log S(\tau)\, d\tau$$

i.e. in the limit as $n \to \infty$

$$Q \to \exp\left(\frac{1}{T}\int_0^T \log S(\tau)\, d\tau\right).$$

Now Q is clearly the GEOMETRIC average of S. So the option is a average strike with a GEOMETRIC AVERAGE.

(b) Now we have

$$V = F(\theta, t), \quad \theta = \frac{I + (T-t)\log S}{T}$$

$$
\begin{aligned}
V_t &= F_\theta \theta_t + F_t t_t = F_\theta \left(-\frac{1}{T} \log S \right) + F_t \\
V_I &= F_\theta \theta_I + F_t t_I = \frac{1}{T} F_\theta + 0 \\
&= \frac{1}{T} F_\theta \\
V_S &= F_\theta \theta_S + F_t t_S = F_\theta \left(\frac{T-t}{ST} \right) + 0 \\
&= F_\theta \left(\frac{T-t}{ST} \right) \\
V_S S &= \left[F_\theta \left(\frac{T-t}{ST} \right) \right]_S = F_{\theta\theta} \theta_S \left(\frac{T-t}{ST} \right) - F_\theta \left(\frac{T-t}{TS^2} \right) \\
&= F_{\theta\theta} \left[\frac{T-t}{ST} \right]^2 - F_\theta \left[\frac{T-t}{TS^2} \right].
\end{aligned}
$$

Substitute in the equation:-

$$
F_t - \frac{1}{2}(\log S) F_\theta + (\log S) \left(\frac{1}{T} F_\theta \right) - rF + rSF_\theta \left(\frac{T-t}{ST} \right)
$$

$$
+ \frac{1}{2}\sigma^2 S^2 \left[F_{\theta\theta} \left(\frac{T-t}{ST} \right)^2 - F_\theta \left(\frac{T-t}{TS^2} \right) \right] = 0
$$

Now rearranging gives

$$
\begin{aligned}
F_t + \frac{\sigma^2}{2} \frac{(T-t)^2}{T^2} F_{\theta\theta} + F_\theta \left(\frac{r(T-t)}{T} - \frac{1}{2}\sigma^2 \left(\frac{T-t}{T} \right) \right) & \\
-rF &= 0 \\
F_t + \frac{\sigma^2}{2} \frac{(T-t)^2}{T^2} F_{\theta\theta} + \left(\frac{T-t}{T} \right) \left(r - \frac{\sigma^2}{2} \right) F_\theta - rF &= 0 \\
\text{i.e. } \mathrm{F_t + a(t)F_{\theta\theta} + b(t)F_\theta - rF} &= 0
\end{aligned}
$$

Where

$$
a(t) = \frac{\sigma^2}{2} \frac{(T-t)^2}{T^2}, \quad b(t) = \left(\frac{T-t}{T} \right) \left(r - \frac{\sigma^2}{2} \right).
$$

(c) The equation just derived is a PDE, but it only has 2 independent variable, as opposed to original equation which has 3 (t, I and S). The new problem is thus ‘one dimension easier’ to solve.

If the payoff depends on S as well as I, then this transformation cannot work, for at $t = T$ (i.e. at expiry when the payoff takes place)

$$\theta = \frac{I + (T - t)\log S}{T} = \frac{I}{T}$$

Thus $V = f(I/T, t)$ and cannot therefore be made to depends upon S.