Question

A PATH-DEPENDENT European option with expiry T is a European option whose payoff is dependent on $S(T)$ and the quantity

$$\int_0^T f(S(\tau), \tau)\, d\tau$$

where f is a given function of S and t. By defined a new independent variable

$$I = \int_0^t f(S(\tau), \tau)\, d\tau$$

show that the stochastic differential equation satisfied by I is

$$dI = f(S, t)dt.$$

Use this result and an appropriate form of Ito's lemma to show that the partial differential equation satisfied by such options is

$$V - t + f(S, t)V_I + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.$$

Now consider the EUROPEAN AVERAGE STRIKE option where

$$f(S, t) = S(t).$$

Show that the partial differential equation is satisfied by solutions of the form

$$V = SU(\eta, t) \qquad (\eta = I/S)$$

provided that U satisfies a given partial differential equation (which you should derive).

Answer
For the PATH-DEPENDENT option we have a payoff dependent on S AND $\int_0^T f(S(\tau),\tau)\,d\tau$.
Define the new indpt variable

$$I = \int_0^t f(S(\tau),\tau)\,d\tau$$

Then

$$\begin{aligned} I(t+dt) &= I + dI = \int_0^{t+dt} f(S(\tau),\tau)\,d\tau \\ &= \int_0^t f(S(\tau),\tau)\,d\tau + \int_t^{t+dt} f(S(\tau),\tau)\,d\tau \\ &= I + f(S(t),t)dt \\ \Rightarrow dI &= f(S,t)dt \end{aligned}$$

Now Ito's lemma will be exactly the same as normal, save for the addition of an $f(S,t)V_I$ term (since dI has no random component).

$$dV = \sigma S V_S dX + \left(\frac{1}{2}\sigma^2 S_{SS}^V + rSV_S + V_t + f(S,t)V_I\right) dt.$$

Now consider a portfolio $\Pi = V - \Delta S$ as usual.
We have

$$\begin{aligned} d\Pi &= dV - \Delta dS \\ &= -\Delta(\sigma S dX + rSdt) + dV \\ &= (\sigma S V_S - \Delta\sigma S)dX \\ &\quad + \left(\frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S + V_t + f(S,t)V_I - rS\Delta\right) dt \end{aligned}$$

As usual, eliminate randomness by choosing $\Delta = V_S$

$$\Rightarrow d\Pi = \left(\frac{1}{2}\sigma^2 S^2 V_S + V_t + f(S,t)V_I\right) dt = r\Pi dt$$

by using the usual arbitrage argument that since $d\Pi$ is riskless, it must grow at the risk free rate.

$$\Rightarrow \frac{1}{2}\sigma^2 S^2 V_{SS} + V_t + f(S,t)V_I = r(V - SV_S)$$

$$\Rightarrow V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + f(S,t)V_I + rSV_S - rV = 0$$

- Black-Scholes for a path dependent option.
Now consider $V = SU(\eta, t) \quad \eta = I/S$
We have

$$\begin{aligned}
V_t &= SU_t \\
V_I &= SU_\eta / S = U_\eta \\
V_S &= U + SU_\eta \left(-\frac{I}{S^2}\right) = U - \left(\frac{I}{S}\right) U_\eta \\
V_{SS} &= U_S + \left(\frac{I}{S^2}\right) U_\eta - \left(\frac{I}{S}\right) U_{\eta\eta} \left(-\frac{I}{S^2}\right) \\
&= U_\eta \left(-\frac{I}{S^2}\right) + \left(\frac{I}{S^2}\right) U_\eta + \left(\frac{I^2}{S^2}\right) U_{\eta\eta} \\
&= \left(\frac{I^2}{S^3}\right) U_{\eta\eta}
\end{aligned}$$

So in the previous equation we get

$$\begin{aligned}
SU_t + \frac{1}{2}\sigma^2 S^2 \frac{S^2 I^2}{S^3} U_{\eta\eta} + SU_\eta + rS\left(U - \frac{I}{S} U_\eta\right) - rSU &= 0 \\
U_t + \frac{1}{2}\sigma^2 \frac{I^2}{S^2} U_{\eta\eta} + U_\eta \left[1 - \frac{rI}{S}\right] + rU - rU &= 0 \\
\Rightarrow U_t + \frac{1}{2}\sigma^2 \eta^2 U_{\eta\eta} + [1 - r\eta] U_\eta &= 0
\end{aligned}$$