Question

(a) If A is a constant, show that $V = AS$ is a solution of the Black-Scholes equation. What "option" has this value?

Call options with strike E and expiry T are to be written on a share that pays a dividend. The structure of the dividend payment is as follows; a single payment with yield y (so that the amount received by the holder is yS) will be made at a time t_d, T. The fair value of such options is denoted by $C(S, T; E, T)$.

(b) Explain why the option price remains continuous as the dividend date is crossed, but the share price drops from S to $(1-y)S$. Give details of the arbitrage possibilities that would exist if S did not jump to $(1-y)S$ across $t = t_d$.

(c) Let $V(S, t; E, T)$ denote the fair (Black-Scholes) price for a Call option on a share that pays no dividends with strike E and expiry T. Show that

$$C(S, t; E, T) = \begin{cases} V(S, t; E, T) & (t_d \leq t \leq T) \\ (1 - y)V(S, t; E/(1 - y), T) & () \leq t \leq t_d) \end{cases}$$

(d) Using a financial argument or otherwise, determine whether

$C(S, t; E, T)$ is larger or smaller than $V(S, t; E, T)$.

Answer

(a) Consider $V = AS$ in Black-Scholes.

$$\begin{aligned} \Rightarrow 0 + \frac{1}{2}\sigma^2 S^2(0) + rSA - rAS &= 0 \\ \Rightarrow rSA - rSA &= 0 \end{aligned}$$

So that $V = AS$ clearly satisfies Black-Scholes. The "option" with this value consists simply of A shares in the underlying - which eventually had value AS.

(b) Underlying pays yS at $t_d < T$: strike$= E$, expiry$= T$.

The option itself pays no dividends, and so there are no abrupt changes in its value as $t = t_d$ is crossed and it is therefore continuous.

Now suppose that S DID NOT jump to $(1-y)S$ across $t = t_d$. If it jumped to a value $S^+ > (1-y)S$ then we arbitrage by short selling S before t_d, pay the dividend qS and buying back straight after t_d.

If $S^+ < (1-y)S$ then buy the asset before t_d, collect the dividend and then sell $\Rightarrow$ risk free profit.

(c) Now $V(S, t; E, T)$ is the value of a call option on a share paying no dividends. Since C is continuous we have

$$C(S^-, t_d^-) = C(S^+, t_d^+)$$

but

$$\begin{aligned} S^+ &= (1-y)S^- \\ \Rightarrow C(S^-, t_d^-) &= c((1-y)S^-, t_d^+) \end{aligned}$$

i.e. the required jump condition is

$$C(S, t_d^-) = C((1-y)S, t_d^+).$$

Now for $t > t_d$ C satisfies

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC = 0$$

with $C(S,T) = \max(\mathrm{S} - \mathrm{E}, 0)$. By <u>definition</u> the solution to this is

$$V(S, t; E, T).$$

Thus

$$C(S,t;E,T) = V(S,t;E,T) \quad (t_d \le t \le T).$$

Now we can use the jump condition:- at t_d

$$C(S,t_d^-;E,T) = C((1-y)S,t_d^+) = V((1-y)S,t_d^+;E,T)$$

Now as we saw in the first part of the question that AS is a solution of Black-Scholes for any A, so certainly $V((1-y)S,t;E,T)$ is.

What does $V((1-y)S,t)$ do at expiry?

Well

$$\begin{aligned} V((1-y)S,T) &= \mathrm{max((1-y)S-E,0)} \\ &= (1-y)\mathrm{max(S-E/(1-y),0)} \end{aligned}$$

i.e. the payoff is the same as $1-y$ calls with a strike $E/(1-y)$.

Thus for $t < t_d$

$$C(S,t;E,T) = (1-y)C(S,t;E/(1-y),T)$$

and so finally

$$C(S,t;E,T) = \begin{cases} V(S,t;E,T) & (T \ge t \ge t_d) \\ (1-y)\mathrm{max(S,t;E/(1-y),T)} & (t_s \ge t) \end{cases}$$

(d) C <u>must</u> be less than V; the option pays no dividend, but the underlying suffers a fall in prices because of the dividend. Its upside potential must therefore be less and so $C < V$