Question

Give a short explanation of what is involved in a FORWARD CONTRACT and explain briefly the major differences between a forward contract and a future. Denoting the fair value of a forward contract by F, show that

$$F = S(t_0)e^{r(T-t_0)}$$

where t_0 is the date on which the forward contract was greed, T is the delivery date, r is the interest rate and S is the price of the underlying asset.

Now suppose that we wish to value a forward contract using the Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.$$

Show that this equation has solutions of the form

$$V(S,t) + AS + Bf(t)$$

where A and B are constants and f is to be determined. By using this solution along with the condition that must be satisfied at $t = T$ show that

$$V(S,t) + S - Fe^{-r(T-t)}$$

and hence again show that

$$F = S(t_0)e^{r(T-t_0)}.$$

An OPTION ON A FUTURE has a value $V(F,t)$ where $F = Se^{-r(T-t)}$. Show from the Black-Scholes equation that V satisfies

$$V_t + \frac{1}{2}\sigma^2 F^2 V_{FF} - rV = 0.$$

Answer

A FORWARD CONTRACT is an agreement between two parties in which one agrees to but a specific asset from the other at a given agreed "forward price" at a specific "delivery date". No money changes hands until the delivery date and at that date, BOTH parties are committed to doing the deal.

A future is just the same as a forward contract, except

(a) rather than being set up between 2 disparate parties, futures are traded on specific exchanges and are subject to the rules and customs of that exchange

(b) the payoff (from whoever to whoever) is evaluated and paid at regular intervals, rather than just at the delivery date.

Now let F denote the fair value of a forward contract. Then if the contract was agreed at t_0 the party who must deliver the asset at T can buy the asset (cost $S(t_0)$) having borrowed the money to do so from the bank.

To borrow $S(t_0)$ for a period $T - t_0$ at interest rate r costs $S(t_0)e^{r(T-t_0)}$ and thus

$$F = S(t_0)e^{r(T-t_0)}$$

Now consider Black-Scholes

$$V - t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.$$

Try for solution

$$V(S,t) = AS + Bf(t)$$

$$\begin{aligned} \Rightarrow Bf' + \frac{1}{2}\sigma^2 S^2(0) + rSA - r(AS + Bf) &= 0 \\ Bf' + rSA - rAS - rBf &= 0 \\ \Rightarrow f' &= rf. \end{aligned}$$

Then $f = ke^{rt}$ where k is a constant (which can obviously be absorbed into B).

Thus

$$V(S,t) = AS + Be^{rt}$$

Now at $r = T$ (expiry) we must have $V(S,T) = S - F$ (the value must be what it's worth now minus what we paid for it.

$$\Rightarrow AS + Be^{eT} = S - F$$

This must be true for ANY $S \Rightarrow A = 1$, $Be^{rT} = -F$

$$\begin{aligned} \Rightarrow B &= -Fe^{-rT} \\ \Rightarrow V(S,t) &= S - Fe^{-r(T-t)} \end{aligned}$$

Now at t_0 the value is 0 (no money is paid until expiry)
$\Rightarrow V(S, t_0) = 0 = S(t_0) - Fe^{-r(T-t_0)}$
and so again

$$F = S(t_0)e^{r(T-t_0)}$$

Now $V = V(F, t)$ where $F = Se^{r(T-t)}$

$$\begin{aligned} \Rightarrow V_t &\rightarrow V_t + F_t V_F = V_t - rFV_F \\ V_S &\rightarrow V_t t_s + V_F F_S = e^{r(T-t)} V_F \\ V_{SS} &\rightarrow e^{2r(T-t)} V_{FF} \end{aligned}$$

So Black-Scholes becomes

$$\begin{aligned} V_t + rFV_F + \frac{\sigma^2}{2} S^2 e^{2r(T-t)} V_{FF} + rSe^{r(T-t)} V_F - rV &= 0 \\ V_t - rFV_F + \frac{\sigma^2}{2} F^2 V_{FF} + rFV_F - rV &= 0 \end{aligned}$$

$$\Rightarrow V - t + \frac{\sigma^2}{2} F^2 V_{FF} - rV = 0$$