QUESTION

(i) Express $\tan x$ as a quotient of trigonometric functions and hence, or otherwise, derive the derivative of $\tan x$ with respect to x.

(ii) State the domain and range of the function $x = \tan_{-1} y$ and sketch its graph.

(iii) Use the identity $1 + \tan^2 x = \sec^2 x$ to show that

$$\frac{d}{dy}(\tan^{-1} y) = \frac{1}{1+y^2}.$$

(iv) Obtain the first and second derivatives with respect to x of the function $u = x\tan^{-1}(x^3)$ and hence verify that u satisfies the equation

$$x^2\frac{d^2u}{dx^2} - 4x\frac{du}{dx} + 4u = -\frac{18x^{10}}{(1+x^6)^2}.$$

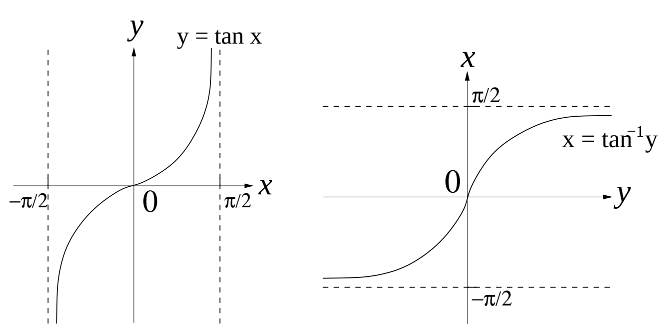
ANSWER

(i) $\tan x = \dfrac{\sin x}{\cos x}$

$$\frac{d}{dx}(\tan x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

(ii) If $x = \tan^{-1} y$, then $y = \tan x$.



Domain $-\infty < y < \infty$ Range $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(iii) $x = \tan^{-1} y,\ y = \tan x$

Therefore $\dfrac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$

Therefore $\dfrac{dx}{dy} = \dfrac{1}{1+y^2}$, or $\dfrac{d}{dy}(\tan^{-1} y) = \dfrac{1}{1+y^2}$

(iv) $u = x \tan^{-1}(x^3)$

$\dfrac{du}{dx} = \tan^{-1}(x^3) + x\left\{\dfrac{1}{1+(x^3)^2}.3x^2\right\} = \tan^{-1}(x^3) + \dfrac{3x^3}{1+x^6}$

$$\begin{aligned}\frac{d^2u}{dx^2} &= \frac{1}{1+(x^3)^2}.3x^2 + \frac{(1+x^6)9x^2 - 3x^3(6x^5)}{(1+x^6)^2}\\ &= \frac{3x^2}{1+x^6} + \frac{9x^2(1-x^6)}{(1+x^6)^2}\\ &= \frac{3x^2\left\{1+x^6+3-3x^6\right\}}{(1+x^6)^2}\\ &= \frac{3x^2(4-2x^6)}{(1+x^6)^2}\end{aligned}$$

i.e. $\dfrac{d^2u}{dx^2} = \dfrac{6x^2(2-x^6)}{(1+x^6)^2}$

Hence,

$$\begin{aligned}&x^2\frac{d^2u}{dx^2} - 4x\frac{du}{dx} + 4u\\ =\ &\frac{6x^4(2-x^6)}{(1+x^6)^2} - 4x\left\{\tan^{-1}(x^3) + \frac{3x^3}{1+x^6}\right\} + 4x\tan^{-1} x^3\\ =\ &\frac{6x^4(2-x^6) - 12x^4(1+x^6)}{(1+x^6)^2}\\ =\ &\frac{6x^4(2-x^6-2-2x^6)}{(1+x^6)^2}\\ =\ &\frac{6x^4(-3x^6)}{(1+x^6)^2}\\ =\ &-\frac{18x^{10}}{(1+x^6)^2}\end{aligned}$$