QUESTION

(a) Obtain the two complex roots (z_1 and z_2) of the quadratic equation

$$(1+j)z^2 - 2jz - (1-j) = 0$$

and calculate

(i) $z_1^2 + z_2^2$, (ii) $\dfrac{1}{z_1} + \dfrac{1}{z_2}$.

(b) Find the solution of the differential equation

$$x\frac{dx}{dt} = (1+x^2)t^2e^{3t},$$

which satisfies $x = 0$ when $t = 0$.

ANSWER

(a) $(1+j)z^2 - 2jz - (1-j) = 0$

$$\begin{aligned} z &= \frac{-(-2j) \pm \sqrt{(-2j)^2 - 4(1+j)(-1+j)}}{2(1+j)} \\ &= \frac{2j \pm \sqrt{-4 - 4(-1+j-j+j^2)}}{2(1+j)} \\ &= \frac{2j \pm \sqrt{4}}{2(1+j)} \\ &= \frac{2j \pm 2}{2(1+j)} \\ &= \frac{\pm 1 + j}{1+j} \end{aligned}$$

With the positive sign, $z_1 = \dfrac{1+j}{1+j} = 1$

With the negative sign, $z_2 = \left(\dfrac{-1+j}{1+j}\right)\left(\dfrac{1-j}{1-j}\right) = \dfrac{-1+j+j-j^2}{1-j^2} = \dfrac{2j}{1+1} = j$

(i) $z_1^2 + z_2^2 = 1^2 + j^2 = 1 - 1 = 0$

(ii) $\dfrac{1}{z_1} + \dfrac{1}{z_2} = \dfrac{1}{1} + \dfrac{1}{j} = 1 + \dfrac{1(-j)}{j(-j)} = 1 - \dfrac{j}{1} = 1 - j$

(b) $x\dfrac{dx}{dt} = \left(1 + x^2\right) t^2 e^{3t}; \quad x(0) = 0$

$\displaystyle\int \frac{x}{1+x^2}\,dx = \int t^2 e^{3t}\,dt.$

Therefore

$$\begin{aligned}
\frac{1}{2}\ln(1+x^2) &= t^2\frac{e^{3t}}{3} - \int \frac{e^{3t}}{3}(2t)\,dt \\
&= \frac{1}{3}t^2e^{3t} - \frac{2}{3}\left\{t\frac{e^{3t}}{3} - \int \frac{e^{3t}}{3}.1\,dt\right\} \\
&= \frac{1}{3}t^2e^{3t} - \frac{2}{9}te^{3t} + \frac{2}{9}\left(\frac{e^{3t}}{3}\right) + c
\end{aligned}$$

$x(0) = 0 \Rightarrow \dfrac{1}{2}\ln 1 = 0 - 0 + \dfrac{2}{27}(1) + c, \ \ c = -\dfrac{2}{27}$

Therefore

$$\ln(1+x^2) = \frac{2}{3}t^2e^{3t} - \frac{4}{9}te^{3t} + \frac{4}{27}e^{3t} - \frac{4}{27}$$

or

$$x^2 = \exp\left(\frac{2}{3}t^2e^{3t} - \frac{4}{9}te^{3t} + \frac{4}{27}e^{3t} - \frac{4}{27}\right) - 1$$

(Take the square root for x.)