Question

If f is measurable prove that for all $a, b\epsilon\mathbf{R}$ $\{x|a \leq f(x) < b\}$ is measurable. Is the converse of this result true?

Answer

$\{x|a \leq f(x) < b\} = \{x|f(x) < b\} \cap \{x|f(x) \geq a\}$

The converse is not true, for example, let $\mathbf{R^n_+}$ be the half space $x_1 > 0$. Let A be a non-measurable subset of $\mathbf{R^n_+}$. Define $f : \mathbf{R^n} \to \mathbf{R}^*$ by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \not\epsilon \mathbf{R^n_+} \\ +\infty & \text{if} \quad x\epsilon A \\ -\infty & \text{if} \quad x\epsilon\mathbf{R^n_+} - A \end{cases}$$

Then for all $a, b\epsilon\mathbf{R}$, $\{x|a \leq f(x) < b\}$ is either ϕ or the complement of $\mathbf{R^n_+}$, both of which are measurable. However $\{x|f(x) > 0\} = A$ which is non-measurable.