Vector Calculus
Grad, Div and Curl Identities

Question
It is given that ϕ and ψ are scalar fields and $\underline{F}$ and $\underline{G}$ are vector fields. They are all assumed to be smooth functions. Prove the following identity

$$\nabla \times (\nabla \times \underline{F}) = \nabla(\nabla \bullet \underline{F}) - \nabla^2 \underline{F}$$

Answer
For $\nabla \times (\nabla \times \underline{F})$, the first component is

$$\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)$$

$$= \frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F - 3}{\partial z \partial x}.$$

For $\nabla(\nabla \bullet \underline{F})$, the first component is

$$\frac{\partial}{\partial x}\nabla \bullet \underline{F} = \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}$$

For $-\nabla^2 \underline{F}$, the first component is

$$-\nabla^2 F_1 = -\frac{\partial^2 F_1}{\partial x^2} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}.$$

It can be seen that the first components of the identity agree, as do the other components by symmetry. So

$$\nabla \times (\nabla \times \underline{F}) = \nabla(\nabla \bullet \underline{F}) - \nabla^2 \underline{F}$$