QUESTION

(a) If $z = x + jy$, where x and y are real numbers, express the complex number $w = \frac{1}{1+z}$ in the form $u + jv$, where u and v are real numbers.

Verify that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$.

(b) Find the general solution to the differential equation

$$\frac{dx}{dt} = e^{2x}t^2\cos(2t).$$

ANSWER

(a)

$$\begin{aligned} w &= \frac{1}{1+z} \\ &= \frac{1}{1+x+jy} \\ &= \frac{1}{(1+x+jy)}\left(\frac{1+x-jy}{1+x-jy}\right) \\ &= \frac{1+x-jy}{(1+x)^2 - j^2y^2} \\ &= \frac{1+x-jy}{(1+x)^2+y^2} \\ &= u+jv \end{aligned}$$

$$u = \frac{1+x}{(1+x)^2+y^2}, \qquad v = \frac{-y}{(1+x)^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\{(1+x)^2+y^2\}(1) - (1+x)\{2(1+x)\}}{\{(1+x)^2+y^2\}^2} = \frac{y^2-(1+x)^2}{\{(1+x)^2+y^2\}^2}$$

$$\frac{\partial v}{\partial y} = \frac{\{(1+x)^2+y^2\}(-1) - (-y)(2y)}{\{(1+x)^2+y^2\}^2} = \frac{y^2-(1+x)^2}{\{(1+x)^2+y^2\}^2}$$

thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

(b) $\dfrac{dx}{dt} = e^{2x}t^2\cos(2t)$

$\displaystyle\int e^{-2x}\,dx = \int t^2\cos(2t)\,dt.$

Therefore

$$\begin{aligned}
\frac{e^{-2x}}{-2} &= t^2\frac{\sin(2t)}{2} - \int \frac{\sin(2t)}{2}(2t)\,dt \\
&= \frac{1}{2}t^2\sin(2t) - \left\{t\left(-\frac{\cos(2t)}{2}\right) - \int -\frac{\cos(2t)}{2}.1\,dt\right\} \\
&= \frac{1}{2}t^2\sin(2t) + \frac{1}{2}t\cos(2t) - \frac{1}{2}\frac{\sin(2t)}{2} + c \\
&= \frac{1}{2}t^2\sin(2t) + \frac{1}{2}t\cos(2t) - \frac{1}{4}\sin(2t) + c \\
e^{-2x} &= -t^2\sin(2t) - t\cos(2t) + \frac{1}{2}\sin(2t) + c \\
-2x &= \ln\left\{\left(\frac{1}{2} - t^2\right)\sin(2t) - t\cos(2t) + c\right\} \\
x &= \ln\left\{\left(\frac{1}{2} - t^2\right)\sin(2t) - t\cos(2t) + c\right\}^{-\frac{1}{2}}
\end{aligned}$$